

MECHANICS OF CONTINUOUS MEDIA IN (\bar{L}_n, g) -SPACES.

I. Introduction and mathematical tools

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Abstract

Basic notions and mathematical tools in continuum media mechanics are recalled. The notion of exponent of a covariant differential operator is introduced and on its basis the geometrical interpretation of the curvature and the torsion in (\bar{L}_n, g) -spaces is considered. The Hodge (star) operator is generalized for (\bar{L}_n, g) -spaces. The kinematic characteristics of a flow are outline in brief.

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1 Introduction

1.1 Differential geometry and space-time geometry

In the last years, the evolution of the relations between differential geometry and space-time geometry has made some important steps toward applications of more comprehensive differential-geometric structures in the models of space-time than these used in (pseudo) Riemannian spaces without torsion (V_n -spaces) [1], [2].

1. Recently, it has been proved that every differentiable manifold with one affine connection and metrics $[(L_n, g)$ -space] [3], [4] could be used as a model for a space-time. In it, the equivalence principle (related to the vanishing of the components of an affine connection at a point or on a curve in a manifold) holds [5] ÷ [11], [12]. Even if the manifold has two different (not only by sign) connections for tangent and co-tangent vector fields $[(\bar{L}_n, g)$ -space] [13], [14] the principle of equivalence is fulfilled at least for one of the two types of vector fields [15]. On this grounds, every free moving spinless test particle in a suitable basic system (frame of reference) [16], [17] will fulfil an equation identical with the equation for a free moving spinless test particle in the Newtons mechanics or in the special relativity. In (\bar{L}_n, g) - and (L_n, g) -spaces, this equation could be the auto-parallel equation [different from the geodesic equation in contrast to the case of (pseudo) Riemannian spaces without torsion (V_n -spaces)].

2. There are evidences that (L_n, g) - and (\bar{L}_n, g) -spaces can have similar structures as the V_n -spaces for describing dynamical systems and the gravitational interaction. In such type of spaces one could use Fermi-Walker transports [18] ÷

[20] conformal transports [21], [22] and different frames of reference [17]. All these notions appear as generalizations of the corresponding notions in V_n -spaces. For instance, in (L_n, g) - and (\overline{L}_n, g) -spaces a proper non-rotating accelerated observer's frame of reference could be introduced by analogy of the same type of frame of reference related to a Fermi-Walker transport [23] in the Einstein theory of gravitation (ETG).

3. All kinematic characteristics related to the notion of relative velocity [24], [25] as well as the kinematic characteristics related to the notion of relative acceleration have been worked out for (L_n, g) - and (\overline{L}_n, g) -spaces without changing their physical interpretation in V_n -spaces [26]. Necessary and sufficient conditions as well as only necessary and only sufficient conditions for vanishing shear, rotation and expansion acceleration induced by the curvature are found [26]. The last results are related to the possibility of building a theoretical basis for construction of gravitational wave detectors on the grounds of gravitational theories over (L_n, g) and (\overline{L}_n, g) -spaces.

Usually, in the gravitational experiments the measurements of two basic objects are considered [27]:

(a) The relative velocity between two particles (or points) related to the rate of change of the length (distance) between them. The change of the distance is supposed to be caused by the gravitational interaction.

(b) The relative accelerations between two test particles (or points) of a continuous media. These accelerations are related to the curvature of the space-time and supposed to be induced by a gravitational force.

Together with the accelerations induced by the curvature in V_n -spaces accelerations induced by the torsion would appear in U_n -spaces, as well as by torsion and by non-metricity in (L_n, g) - and (\overline{L}_n, g) -spaces.

In particular, in other models of a space-time [different from the (pseudo) Riemannian spaces without torsion] the torsion has to be taken into account if we consider the characteristics of the space-time.

4. On the one hand, until now, there are a few facts that the torsion could induce some very small and unmeasurable effects in quantum mechanical systems considered in spaces with torsion [28]. At the same time, there are no evidences that the model's descriptions of interactions on macro level should include the torsion as a necessary mathematical tool. From this (physical) point of view the influence of the torsion on dynamical systems could be ignored since it could not play an important role in the description of physical systems in the theoretical gravitational physics. On the other hand, from mathematical point of view (as we will try to show in this paper), the role of the torsion in new theories for description of dynamical systems could be important and not ignorable.

1.2 Differential geometry and continuous media mechanics

1. In many considerations of the behavior of materials and physical systems the motion of atoms and molecules in it is ignored and a material (physical system) is considered as a whole subject. The molecular structure is not investigated and the assumption is made that the matter is continuously distributed in the volume occupied by it. This conception for a continuous media appears as the basic conception of the mechanics of continuous media (continuum media mechanics). In the range of the confined conditions, where the basic conception could be valid, it could be used for description of rigid bodies, fluids, and gases. Moreover, it has been shown [29], [30] that every classical (non-quantized) field theory could be considered as a theory of a continuous media. This fact gives rise to generalization and application of the structures of continuous media mechanics to different mathematical models of classical field theories and especially in theories of gravitation.

2. In Einstein's theory of gravitation the relativistic continuous media mechanics has been worked out by Synge [31], Lichnerowicz [32] and especially by Ehlers [33]. Most of the notions of the classical mechanics of continuous media have been generalized and applied to finding out invariant characteristics of Einstein's field equations as well as to physical interpretation of formal results in the gravitational theory. Hydrodynamics has also been considered as an essential part of a gravitational theory [33].

3. Notions of relativistic continuous media mechanics have been generalized for spaces with affine connections and metrics [26] ÷ [30]. Conditions have been found for these types of spaces admitting special vector fields with vanishing kinematic characteristics related to the notions of mechanics of continuous media. Nevertheless, the physical interpretation of the generalized notions is in some cases not so obvious as in the classical continuous mechanics and requires additional considerations. Usually, this is done by the use of congruencies (families of non-intersecting curves), interpreted as world lines describing the motion of matter elements in an infinitesimal region of a flow [33], [24]. The general considerations include determination of the kinematic characteristics of a flow such as the velocity, the relative velocity and relative acceleration between the material points (elements) as well as the influence of the geometric structure (curvature, torsion) on these kinematic characteristics. An invariant description of a continuous media is related to the notions of covariant derivatives along curves, to covariant and Lie differential operators acting on corresponding vector and tensor fields. The acceptance of the hypothesis of a continuous media (continuum) as a basis for mathematical description of the behavior of matter means also that the set of quantities (such as deformation, stress, transports, draggings along etc.) should be expressed by means of (at least) piecewise continuous functions of space and time.

In the present paper, basic notions of continuum media mechanics are introduced and considered in spaces with affine connections and metrics $[(\overline{L}_n, g)$ -spaces]. In Section 2 basic notions and mathematical tools are recalled. The notion of exponent of a covariant differential operator is introduced and on its basis the geometrical interpretation of the curvature and the torsion in (\overline{L}_n, g) -spaces is considered in Section 3 and Section 4. In Section 5 the kinematic characteristics of a flow are considered.

All considerations are given in details (even in full details) for those readers who are not familiar with the considered problems.

Remark. The present paper is the first part of a larger research report on the subject with the title "Contribution to continuous media mechanics in (\overline{L}_n, g) -spaces" and with the following contents:

- I. Introduction and mathematical tools.
- II. Relative velocity and deformations.
- III. Relative accelerations.
- IV. Stress (tension) tensor.

The parts are logically self-dependent considerations of the main topics considered in the report.

2 Basic notions and mathematical tools

For the further considerations we need to recall some well known facts and definitions from continuous media mechanics.

A mathematical model of a continuum (continuous media) is assumed to be a differentiable manifold M with dimension $\dim M = n$. In this continuum all structures, kinematic and dynamic characteristics of a media could be described

and considered by the use of all differential-geometric structures in the differentiable manifold M . The basic notions of continuous media mechanics are related to problems of description of [34]

- (a) deformations
- (b) stresses (tensions)
- (c) relation between stresses (tensions) and deformations
- (d) dynamical reasons for generation of deformations and stresses (tensions).

The notion of motion and flow are used in the sense of instant (in a moment) motion and in the sense of continuous motion. In some cases the notion of flow is used in the sense of motion causing residual deformation. In the theory of fluids (as a part of continuous media mechanics) the notion of flow means continuous motion. We will use the notion of flow (when no other conditions are supposed) in the sense of continuous motion.

The reason for deformation in a media is the motion of its material points to each other. This motion could be expressed in two different ways:

- (a) By means of the change of a vector field ξ (called deformation or deviation vector field) along a vector field u , interpreted as the velocity of the material points, i.e. a deformation is described by means of the covariant derivative $\nabla_u \xi$ of ξ along u .
- (b) By means of the change of the velocity vector field u along the deformation (deviation) vector field ξ , i.e. a deformation is described by means of the covariant derivative $\nabla_\xi u$ of u along ξ .

The vector fields u and ξ could be chosen as tangent vector fields to the coordinates introduced in a media, considered as a differentiable manifold M with $\dim M = n$. For $n \leq 3$ the classical (non-relativistic) continuous media mechanics has been developed. For $n = 4$ the relativistic continuous media mechanics has been worked out. For $n \geq 4$ models could also be worked out describing a continuous media in more sophisticated spaces with affine connections and metrics.

1. There are basic definitions related to the notion of flow in the continuous media mechanics. Let us recall some of them.

Trajectory is a line at which a material point moves during its motion.

Motion is the change of the position of a material point in a differentiable manifold considered as a model of space-time.

Position of a material point is the point of a differentiable manifold identified with the material point.

Space is a differentiable manifold provided with additional structures and considered as a model of a physical space or space-time.

Physical space is a sub space of a space-time considered in many mathematical models of space-time as a sub space orthogonal to the time.

Homogeneous are spaces or matter if they have the same properties at all points of a manifold (considered as a model of a space, space-time, or matter).

Isotropic are spaces or matter with respect to a given property if this property at a point of a manifold is the same for all directions going out of this point.

Anisotropic are spaces or matter with respect to a given property if this property is different for different directions going out of a given point.

Line of a flow is a line with tangential vector at each of its points collinear (or identical) with the velocity of a material point with the corresponding position.

Flow is a congruence (family) of flow's lines.

2. Every motion of material points of a dynamical system is described by the use of a frame of reference [16] related to an observer with velocity vector field u .

The use of a contravariant non-isotropic (non-null) vector field u and its corresponding projective metrics h^u and h_u is analogous to the application of a non-isotropic (time-like) vector field in the s. c. *monad formalism* [(3+1)-decomposition]

in V_4 -spaces for description of dynamical systems in Einstein's theory of gravitation (ETG) [35] ÷ [38].

The contravariant time-like vector field u has been interpreted as a tangential vector field at the world line of an observer determining the frame of reference (the reference frame) in the space-time. By the use of this reference frame a given physical system is observed and described. The characteristics of the vector field determine the properties of the reference frame. Moreover, the vector field is assumed to be an absolute element in the scheme for describing the physical processes, i. e. the vector field is not an element of the model of the physical system. It is introduced as *a priori* given vector field which does not depend on the Lagrangian system. In fact, the physical interpretation of the contravariant non-isotropic vector field u can be related to two different approaches analogous to the method of Lagrange and the method of Euler in describing the motion of liquids in the hydrodynamics [39].

The $(n - 1) + 1$ decomposition (monad formalism) can be related to the method of Euler or to the method of Lagrange on the basis of a frame of reference.

There are at least two types of frame of reference with respect to the motion of material points:

(a) *Proper frame of reference.* A frame of reference of an observer, moving with a material point of the dynamical system, is called proper frame of reference for this point. The position and the velocity of the observer is identified with the position and the velocity of the material point. Its velocity is determined by means of the dynamics of the flow, described by the use of equations of the type of Euler-Lagrange's equations. Solutions of these equations with some initial data describe the motion of the material point (observer) with respect to a given initial time. So, a proper frame of reference is related to the method of Lagrange in continuous media mechanics.

In the method of Lagrange, the object with the considered motion appears as a point (particle) of a flow. The motion of this point is given by means of equations for a vector field u interpreted as the velocity of the particle. The solutions of these equations give the trajectories of the points in the flow as basic characteristics of the physical system. Here, the vector field u appears as an element of the model of the system. It is connected with the motions of the system's elements. Therefore, a Lagrangian system (and respectively its Lagrangian density) could contain as an internal characteristic a contravariant vector field u obeying equations of the type of the Euler-Lagrange equations and describing the evolution of the system.

Lagrangian system is a dynamical system (system obeying physical laws) which characteristics are described by means of an Lagrangian density and its corresponding Euler-Lagrange's equations.

(a₁) *Method of Lagrange.* The vector field u is interpreted as the velocity vector field of a continuum media with an observer co-moving with it. The last assumption means that the velocity vector field of the observer is identical with the velocity vector field of the media where he is situated.

The motion of the observer (his velocity vector field) is determined by the characteristics of the (Lagrangian) system. Its velocity vector field is, on the other side, determined by the dynamical characteristics of the system by means of equations of the type of the Euler-Lagrange equations.

(b) *Non-proper frame of reference.* A frame of reference of an observer, moving in space-time with his own velocity and describing the motion of material points by means of projections of the characteristics of a flow on its own kinematic characteristics of the motion, is called non-proper frame of reference.

The position and the velocity of the observer could be different from the position and the velocity of the observed material point. The equations describing the motions of the observer and the material points could be different. The equation of motion of the observer is assumed to be given.

The non-proper frame of reference is related to the method of Euler in continuous media mechanics.

(b₁) *Method of Euler.* In the method of Euler, the object with the considered motion appears as the model of the continuum media. Instead of the investigation of the motion of every (fixed by its velocity and position) point (particle), the kinematic characteristics in every immovable point in the space are considered as well as the change of these characteristics after moving on to an other space point. The motion is assumed to be described if the vector field u is considered as a given (or known) velocity vector field.

The vector field u is interpreted as the velocity vector field of an observer who describes a physical system with respect to his vector field (his velocity). This physical system is characterized by means of a Lagrangian system (Lagrangian density).

The motion of the observer (his velocity vector field) is given independently of the motion of the considered Lagrangian system.

In a proper frame of reference all kinematic characteristics of the vector field u are at the same time kinematic characteristics of the flow. In a non-proper frame of reference the kinematic characteristics of the vector field u are only characteristics of the frame of reference and not characteristics of the flow.

In our further considerations we will assume the existence of a proper frame of reference in a flow.

3. Continuous media mechanics describes the state of a continuum by the use or two major characteristics (deformation and stress) and relations between them.

(a) A deformation in a continuous media describes the change of the relative positions of the material points in the media. Its invariant (tensor) characteristic is the deformation velocity tensor and the related to it kinematic characteristics called shear velocity tensor, rotation (vortex) velocity tensor, and expansion velocity invariant [26].

(b) A stress (tension) in a continuous media describes the surface forces in the media. A *surface force* is a force acting on a surface element (an element of a surface covering an invariant volume element of continuous media). It is then related to a volume force. A *volume force* is a force acting on a *invariant volume element of the media* identified with the invariant volume element in a differentiable manifold M .

2.1 Covariant derivative of a tensor field along a curve. Exponent of the covariant differential operator

The notion of covariant derivative was introduced as the result of the action of the covariant differential operator on a given tensor field (with finite rank). The covariant differential operator is defined as a differential operator (along a given contravariant vector field) determining the corresponding affine connection. On the other side, a given contravariant vector field u could be considered as a tangent vector field along a parametrized curve $x^i(\tau)$, $i = 1, 2, \dots, n$, $\tau \in \mathbf{R}$, in the manifold M and written in the form

$$u := \frac{d}{dt} = \frac{dx^i}{d\tau} \cdot \partial_i = u^i \cdot \partial_i, \quad u^i = \frac{dx^i(\tau)}{d\tau} \quad . \quad (1)$$

The covariant derivative of a tensor field $V = V^A{}_B \cdot \partial_A \otimes dx^B$, $V \in \otimes^k_l(M)$ along the vector field u

$$\nabla_u V = V^A{}_{B;i} \cdot u^i \cdot \partial_A \otimes dx^B \quad (2)$$

for $u = d/d\tau$ is usually written in the form

$$\nabla_u V = \nabla_{\frac{d}{d\tau}} V := \frac{DV}{d\tau} \quad . \quad (3)$$

At the point of the curve $x^i(\tau_0 = \text{const.}, \lambda_0^a = \text{const.})$ with the parameter $\tau_0 = \text{const.}$ the covariant derivative could be written as

$$(\nabla_u V)_{(\tau_0)} = (\nabla_{\frac{d}{d\tau}} V)_{(\tau_0)} = \left(\frac{DV}{d\tau} \right)_{(\tau_0)}, \quad \lambda_0^a = \text{const.}, \quad (4)$$

where $V_{(x^i(\tau, \lambda_0^a))} = V_{(\tau)}$.

The question arises how can we express the covariant derivative of $V(x^i(\tau)) = V(\tau)$ at the point $x^i(\tau_0 + dt)$, $d\tau = \varepsilon \ll 1$, by means of the covariant derivative at the point $x^i(\tau_0)$.

Exponential mapping of an ordinary differential operator If we wish to express only the components $V^A{}_B(x^i(\tau)) = V^A{}_B(\tau) \in C^r(M)$ at the point $x^i(\tau_0 + \varepsilon)$ by means of the components $V^A{}_B(x^i(\tau_0)) = V^A{}_B(\tau_0)$ we can use the decomposition of $V^A{}_B(\tau_0 + \varepsilon)$ in Taylor row with respect to $V^A{}_B(\tau_0)$ in the form

$$\begin{aligned} V^A{}_B(\tau_0 + \varepsilon) &= V^A{}_B(\tau_0) + \varepsilon \cdot \left(\frac{dV^A{}_B}{d\tau} \right)_{(\tau_0)} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \left(\frac{d^2 V^A{}_B}{d\tau^2} \right)_{(\tau_0)} + \dots = \\ &= \left[\left(1 + \varepsilon \cdot \frac{d}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{d^2}{d\tau^2} + \dots \right) V^A{}_B \right]_{(\tau_0)} = \\ &= \left[\left(\exp\left[\varepsilon \cdot \frac{d}{d\tau}\right] \right) V^A{}_B \right]_{(\tau_0)}. \end{aligned} \quad (5)$$

The operator

$$\exp\left[\varepsilon \cdot \frac{d}{d\tau}\right] = 1 + \varepsilon \cdot \frac{d}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{d^2}{d\tau^2} + \dots \quad (6)$$

is called *exponent of the ordinary differential operator* $\varepsilon \cdot \frac{d}{d\tau}$ [40]. It maps the components $V^A{}_B(\tau_0)$ at a given point $x^i(\tau_0)$ of the curve with a parameter τ_0 to components $V^A{}_B(\tau_0 + d\tau)$ at the point $x^i(\tau)$ of the curve with parameter $\tau = \tau_0 + d\tau$, where $d\tau = \varepsilon \ll 1$. The exponent mapping of $V^A{}_B(\tau_0)$ into $V^A{}_B(\tau_0 + d\tau)$ has as a precondition the conservation (not changing) of the tensor bases $\partial_A \otimes dx^B$ on the curve $x^i(\tau)$. This means that we are considering a tensor field $V = V^A{}_B \cdot \partial_A \otimes dx^B$ in two different neighboring points at the curve assuming that the tensor bases are one and the same at both the points. By the use of the exponent of the ordinary differential operator we can find a geometrical interpretation of the Lie derivative (the commutator of two contravariant vector fields) of a contravariant vector field along an other contravariant vector field [40].

2.1.1 Geometrical interpretation of the Lie derivative of a contravariant vector field along an other contravariant vector field

Let a two parametric congruence of curves $x^i(\tau, \lambda)$ be given. An infinitesimal quadrangle is determined by its apexes $ABDC$ (the points A, B, D, C) with the co-ordinates respectively:

$$\begin{aligned} A &: (\tau_0, \lambda_0) \quad , & B &: (\tau_0, \lambda_0 + d\lambda) \quad , \\ D &: (\tau_0 + k_1 \cdot d\tau, \lambda_0 + k_2 \cdot d\lambda) \quad , & C &: (\tau_0 + d\tau, \lambda_0) \quad , \\ k_1, k_2 &\in \mathbf{R} \quad . \end{aligned} \quad (7)$$

Let us assume that $d\tau = d\lambda = \varepsilon \ll 1$, and $k_1 \cdot d\tau = k_2 \cdot d\lambda = k \cdot \varepsilon \ll 1$. We can now represent the co-ordinates of the points lying at equal parameters ε from the points A, B , and C by the use of the co-ordinates of the point A and the exponent of

the ordinary differential operator. Let these points be C_1 and D_1 with co-ordinates $C_1 : (\tau_0 + d\tau, \lambda_0 + d\lambda)$ on the way ACC_1 and $D_1 : (\tau_0 + d\tau, \lambda_0 + d\lambda)$ on the way ABD_1 . The points A, B, C, D, C_1 , and D_1 will have the co-ordinates expressed by the co-ordinates of the point A as follows:

$$\begin{aligned}
A & : x^i(\tau_0, \lambda_0) \\
B & : x^i(\tau_0, \lambda_0 + d\lambda) = \left[\left(\exp[d\lambda \cdot \frac{d}{d\lambda}] \right) x^i \right]_{(\tau_0, \lambda_0)}, \\
C & : x^i(\tau_0 + d\tau, \lambda_0) = \left[\left(\exp[d\tau \cdot \frac{d}{d\tau}] \right) x^i \right]_{(\tau_0, \lambda_0)}, \\
D & : x^i(\tau_0 + k_1 \cdot d\tau, \lambda_0 + k_2 \cdot d\lambda) = \left(\exp[k_1 \cdot d\tau \cdot \frac{d}{d\tau}] \right) x^i_{(\tau_0, \lambda_0 + d\lambda)} = \\
& = \left[\left(\exp[k_1 \cdot d\tau \cdot \frac{d}{d\tau}] \right) \circ \left(\exp[k_2 \cdot d\lambda \cdot \frac{d}{d\lambda}] \right) x^i \right]_{(\tau_0, \lambda_0)} \\
C_1 & : x^i(\tau_0 + d\tau, \lambda_0 + d\lambda) = \left(\exp[d\lambda \cdot \frac{d}{d\lambda}] \right) x^i_{(\tau_0 + d\tau, \lambda_0)} = \\
& = \left[\left(\exp[d\lambda \cdot \frac{d}{d\lambda}] \right) \circ \left(\exp[d\tau \cdot \frac{d}{d\tau}] \right) x^i \right]_{(\tau_0, \lambda_0)}, \\
D_1 & = x^i(\tau_0 + d\tau, \lambda_0 + d\lambda) = \left(\exp[d\tau \cdot \frac{d}{d\tau}] \right) x^i_{(\tau_0, \lambda_0 + d\lambda)} = \\
& = \left[\left(\exp[d\tau \cdot \frac{d}{d\tau}] \right) \circ \left(\exp[d\lambda \cdot \frac{d}{d\lambda}] \right) x^i \right]_{(\tau_0, \lambda_0)}. \tag{8}
\end{aligned}$$

Now, we have to consider the co-ordinates of the points C_1 and D_1 . For $d\tau = d\lambda = \varepsilon$ we have

$$\begin{aligned}
C_1 & : x^i(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) = \\
& = \left[\left(\exp[\varepsilon \cdot \frac{d}{d\lambda}] \right) \circ \left(\exp[\varepsilon \cdot \frac{d}{d\tau}] \right) x^i \right]_{(\tau_0, \lambda_0)}, \\
D_1 & : x^i(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) = \\
& = \left[\left(\exp[\varepsilon \cdot \frac{d}{d\tau}] \right) \circ \left(\exp[\varepsilon \cdot \frac{d}{d\lambda}] \right) x^i \right]_{(\tau_0, \lambda_0)}. \tag{9}
\end{aligned}$$

By the use of the explicit form of the exponent of the ordinary differential operators $d/d\tau$ and $d/d\lambda$ (identical with the tangent vector fields u and ξ at the congruence of curves), we obtain

$$\begin{aligned}
C_1 & : x^i_{AC}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) = \\
& = \left[\left(1 + \varepsilon \cdot \frac{d}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{d^2}{d\lambda^2} + \dots \right) \circ \left(1 + \varepsilon \cdot \frac{d}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{d^2}{d\tau^2} + \dots \right) x^i \right]_{(\tau_0, \lambda_0)}, \\
D_1 & : x^i_{AB}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) = \\
& = \left[\left(1 + \varepsilon \cdot \frac{d}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{d^2}{d\tau^2} + \dots \right) \circ \left(1 + \varepsilon \cdot \frac{d}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{d^2}{d\lambda^2} + \dots \right) x^i \right]_{(\tau_0, \lambda_0)} \tag{10}
\end{aligned}$$

The difference between the co-ordinates $x^i_{AC}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon)$ and $x^i_{AB}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon)$ can be found as

$$x^i_{AC}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) - x^i_{AB}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon)$$

$$\begin{aligned}
&= \left[\left(\exp\left[\varepsilon \cdot \frac{d}{d\lambda}\right] \right) \circ \left(\exp\left[\varepsilon \cdot \frac{d}{d\tau}\right] \right) x^i \right]_{(\tau_0, \lambda_0)} - \left[\left(\exp\left[\varepsilon \cdot \frac{d}{d\tau}\right] \right) \circ \left(\exp\left[\varepsilon \cdot \frac{d}{d\lambda}\right] \right) x^i \right]_{(\tau_0, \lambda_0)} \\
&= \left[\left(\exp\left[\varepsilon \cdot \frac{d}{d\lambda}\right] \right), \left(\exp\left[\varepsilon \cdot \frac{d}{d\tau}\right] \right) \right] x^i_{(\tau_0, \lambda_0)} . \tag{11}
\end{aligned}$$

Up to the second order of ε , we have

$$\begin{aligned}
C_1 &: x^i_{AC}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) = \\
&= \left[\left[1 + \varepsilon \cdot \left(\frac{d}{d\lambda} + \frac{d}{d\tau} \right) + \varepsilon^2 \cdot \left(\frac{1}{2} \cdot \frac{d^2}{d\lambda^2} + \frac{d}{d\lambda} \circ \frac{d}{d\tau} + \frac{1}{2} \cdot \frac{d^2}{d\tau^2} \right) + O(\varepsilon^3) \right] x^i \right]_{(\tau_0, \lambda_0)} , \\
D_1 &: x^i_{AB}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) = \\
&= \left[\left[1 + \varepsilon \cdot \left(\frac{d}{d\tau} + \frac{d}{d\lambda} \right) + \varepsilon^2 \cdot \left(\frac{1}{2} \cdot \frac{d^2}{d\tau^2} + \frac{d}{d\tau} \circ \frac{d}{d\lambda} + \frac{1}{2} \cdot \frac{d^2}{d\lambda^2} \right) + O(\varepsilon^3) \right] x^i \right]_{(\tau_0, \lambda_0)} , \\
&\quad x^i_{AC}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) - x^i_{AB}(\tau_0 + \varepsilon, \lambda_0 + \varepsilon) \\
&= \left\{ \left[\varepsilon^2 \cdot \left[\frac{d}{d\lambda} \circ \frac{d}{d\tau} - \frac{d}{d\tau} \circ \frac{d}{d\lambda} \right] + O(\varepsilon^3) \right] x^i \right\}_{(\tau_0, \lambda_0)} \\
&= \left[\varepsilon^2 \cdot [\xi \circ u - u \circ \xi] + O(\varepsilon^3) \right]_{(\tau_0, \lambda_0)} \approx \varepsilon^2 \cdot ([\xi, u] x^i)_{(\tau_0, \lambda_0)} . \tag{12}
\end{aligned}$$

On this basis, the relation

$$\begin{aligned}
x^i_{AC} - x^i_{AB} &\approx \varepsilon^2 \cdot \left(\left[\frac{d}{d\lambda} \circ \frac{d}{d\tau} - \frac{d}{d\tau} \circ \frac{d}{d\lambda} \right] x^i \right)_{(\tau_0, \lambda_0)} = \\
&= \varepsilon^2 \cdot ([\xi, u] x^i)_{(\tau_0, \lambda_0)} = \varepsilon^2 \cdot [(\mathcal{L}_\xi u) x^i]_{(\tau_0, \lambda_0)} \tag{13}
\end{aligned}$$

follows.

Therefore, up to the second order of ε , the Lie derivative $\mathcal{L}_\xi u = [\xi, u] = -\mathcal{L}_u \xi$ could be interpreted as a measure for the difference between the co-ordinates of two points C_1 and D_1 lying at equal distances (parameters) from a starting point A . This means that if a (material) point is moving from a point A across a point B to a point D_1 (with equal distances ε from point A to point B and from point B to point D_1), it could not meet a point, moving from a point A across a point C to a point C_1 (with equal distances ε between point A and point C and from point C to point C_1), if the Lie derivative $\mathcal{L}_\xi u = -\mathcal{L}_u \xi$ of the tangent vector ξ along u or vice versa is not equal to zero. If $\mathcal{L}_\xi u = -\mathcal{L}_u \xi = 0$ then both the points C_1 and D_1 will coincide and the points, moving on the different paths ACC_1 and ABD_1 will meet at one and the same point $D_1 \equiv C_1$ closing the quadrangle $AB(C_1 \equiv D_1)C$. In the opposite case, ABC_1D_1C will construct a pentagon.

On the basis of the relation

$$\mathcal{L}_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) , \tag{14}$$

under the conditions for parallel transport of ξ along u and u along ξ , i.e. under the conditions $\nabla_u \xi = 0$ and $\nabla_\xi u = 0$, it follows that

$$\mathcal{L}_\xi u = -T(\xi, u) . \tag{15}$$

The last relation is used for finding another geometric interpretation of the torsion vector $T(\xi, u)$ and respectively of the torsion tensor T . Under the conditions $\nabla_u \xi = 0$ and $\nabla_\xi u = 0$ the torsion vector (and the Lie derivative) appear as a measure for the non-existence of a closed infinitesimal quadrangle with equal sides. This fact is used in theories of crystals for description of the defects in crystal's cells.

2.1.2 Exponent of the covariant differential operator

The covariant operator $\frac{D}{d\tau}$ is a generalization of the differential operator $\frac{d}{d\tau}$ along the curve $x^i(\tau)$ for the case when not only the components $V^A{}_B(\tau)$ of the tensor field $V(\tau) = V^A{}_B(\tau) \cdot \partial_A(\tau) \otimes dx^B(\tau)$ are changing along the curve but also its tensor bases $\partial_A(\tau) \otimes dx^B(\tau)$ are also changing along the curve in accordance with the relations

$$\begin{aligned} \frac{D}{d\tau}(\partial_A \otimes dx^B) &= \nabla_u(\partial_A \otimes dx^B) = (\Gamma_{Ai}^C \cdot \partial_C \otimes dx^B + \partial_A \otimes P_{Di}^B \cdot dx^D) \cdot u^i = \\ &= u^i \cdot (\Gamma_{Ai}^C \cdot g_D^B \cdot \partial_C \otimes dx^D + P_{Di}^B \cdot g_A^C \cdot \partial_C \otimes dx^D) = \\ &= u^i \cdot (\Gamma_{Ai}^C \cdot g_D^B + P_{Di}^B \cdot g_A^C) \cdot \partial_C \otimes dx^D = \\ &= \frac{dx^i}{d\tau} \cdot (\Gamma_{Ai}^C \cdot g_D^B + P_{Di}^B \cdot g_A^C) \cdot \partial_C \otimes dx^D, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Gamma_{Aj}^B &= -S_{Am}{}^{Bi} \cdot \Gamma_{ij}^m, \quad A = j_1 \dots j_l, \quad B = i_1 \dots i_l, \\ S_{Am}{}^{Bi} &= -\sum_{k=1}^l g_{j_k}^i \cdot g_m^{i_k} \cdot g_{j_1}^{i_1} \cdot g_{j_2}^{i_2} \dots g_{j_{k-1}}^{i_{k-1}} \cdot g_{j_{k+1}}^{i_{k+1}} \dots g_{j_l}^{i_l}, \\ P_{Aj}^B &= -S_{Am}{}^{Bi} \cdot P_{ij}^m, \\ g_B^A &= g_{i_1}^{j_1} \dots g_{i_{m-1}}^{j_{m-1}} \cdot g_{i_m}^{j_m} \cdot g_{i_{m+1}}^{j_{m+1}} \dots g_{i_l}^{j_l}. \end{aligned}$$

Only if the components Γ_{jk}^i and P_{jk}^i of the contravariant and covariant affine connections Γ and P respectively vanish along the curve, then

$$\frac{D}{d\tau}(\partial_C \otimes dx^D) = 0 \quad (17)$$

and the tensor bases do not change. Therefore, if we consider the tensor $V(\tau_0 + d\tau)$ at the point $x^i(\tau_0 + d\tau) = x^i(\tau_0 + \varepsilon)$ of the curve and wish to compare it with the tensor $V(\tau_0)$ at the point $x^i(\tau_0)$ of the same curve in a space with a contravariant affine connection Γ and a covariant affine connection P we should change the operator $\frac{d}{d\tau}$ with the operator $\frac{D}{d\tau}$ in the representation of $V(\tau_0 + \varepsilon) = V_{(\tau_0 + \varepsilon)}$ by the use of $V(\tau_0) = V_{(\tau_0)}$ and its covariant derivatives along the curve $x^i(\tau)$, i.e.

$$\begin{aligned} V_{(\tau_0 + \varepsilon)} &= V_{(\tau_0)} + \varepsilon \cdot \left(\frac{DV}{d\tau} \right)_{(\tau_0)} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \left(\frac{D^2V}{d\tau^2} \right)_{(\tau_0)} + \dots = \\ &= \left[\left(1 + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \dots \right) V \right]_{(\tau_0)} = \\ &= \left[\left(\exp\left[\varepsilon \cdot \frac{D}{d\tau} \right] \right) V \right]_{(\tau_0)}. \end{aligned} \quad (18)$$

The operator

$$\exp\left[\varepsilon \cdot \frac{D}{d\tau} \right] = 1 + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \dots \quad (19)$$

could be called *exponent of the covariant differential operator* $\varepsilon \cdot \frac{D}{d\tau}$. It maps the tensor $V_{(\tau_0)}$ at the point $x^i(\tau_0)$ of the curve $x^i(\tau)$ to the tensor $V_{(\tau_0 + \varepsilon)}$ at the point $x^i(\tau_0 + \varepsilon)$ of the same curve as

$$V_{(\tau_0 + \varepsilon)} = \left[\left(\exp\left[\varepsilon \cdot \frac{D}{d\tau} \right] \right) V \right]_{(\tau_0)} \quad (20)$$

under taking into account the change of the components of the tensor in a (co-ordinate) basis as well as the change of the tensor bases along the curve at two neighboring points of the curve. By the use of the exponent of the covariant differential operator we can compare two tensors at two neighboring points of a curve at one of its points.

The expression

$$\left[\left(\exp \left[\varepsilon \cdot \frac{D}{d\tau} \right] \right) V \right]_{(\tau=\tau_0+\varepsilon)} = V_{(\tau=\tau_0)} + \varepsilon \cdot \left(\frac{DV}{d\tau} \right)_{(\tau=\tau_0)} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \left(\frac{D^2V}{d\tau^2} \right)_{(\tau=\tau_0)} + \dots \quad (21)$$

could be called *covariant Taylor row*.

A co-ordinate tensor basis $\partial_A \otimes dx^B$ at the point $x^i(\tau_0 + d\tau)$ could be expressed by means of the tensor basis $\partial_A \otimes dx^B$ at the point $x^i(\tau_0)$ of the curve $x^i(\tau)$ as

$$(\partial_A \otimes dx^B)_{(\tau=\tau_0+d\tau)} = \left[\left(\exp \left[d\tau \cdot \frac{D}{d\tau} \right] \right) (\partial_A \otimes dx^B) \right]_{(\tau=\tau_0)} .$$

Now, we can determine the covariant derivative $\nabla_u V$ of a tensor field $V \in \otimes^k_l(M)$ along a curve $x^i(\tau)$ with tangential vector $u = d/d\tau$ at a given point $x^i(\tau = \tau_0)$ as

$$\nabla_u V_{(\tau_0)} = \left(\frac{DV}{d\tau} \right)_{(\tau_0)} = \lim_{d\tau \rightarrow 0} \frac{V_{(\tau_0+d\tau)} - V_{(\tau_0)}}{d\tau} , \quad d\tau = \varepsilon \ll 1 . \quad (22)$$

By the use of the covariant derivative along a curve and the exponent of the covariant differential operator we can find the geometrical interpretation of the curvature tensor as well as the geometrical interpretation of the torsion vector and the torsion tensor.

2.2 Hodge (star) operator in spaces with affine connections and metrics

In many problems of theoretical physics the full anti-symmetric covariant tensor fields (differential forms) are used. The transition from one differential form of rank $p \leq n$ to a differential form of rank $n - p$, where n is the dimension of the differentiable manifold M over which differential forms are defined, is related to the existence of a map called *Hodge or star operator* [?], [?]. Its explicit action is given in E_n (Euclidean spaces) and V_n -spaces, as well as in spaces with one affine connection and metrics. Its generalization for (\overline{L}_n, g) -spaces requires some additional considerations. Usually the Hodge (star) operator is constructed by means of the permutation (Levi-Civita) symbols. It maps a full covariant anti-symmetric tensor of rank $(0, p) \equiv {}^a \otimes_p(M) \equiv \Lambda^p(M)$ in a full covariant anti-symmetric tensor of rank $(0, n - p) \equiv {}^a \otimes_{n-p}(M) \equiv \Lambda^{n-p}(M)$, where $\dim M = n$.

2.2.1 Definition of the Hodge (star) operator

Let ${}_a A := A_{[i_1 \dots i_k]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}$ be a full covariant anti-symmetric tensor field of rank k , i.e. ${}_a A \in \Lambda^k(M)$. Let ${}_a \overline{A} := A^{[j_1 \dots j_k]} \cdot \partial_{j_1} \wedge \dots \wedge \partial_{j_k}$ be the corresponding full contravariant anti-symmetric tensor field of rank k , i.e. ${}_a \overline{A} \in {}_a \otimes^k(M)$. ${}_a \overline{A}$ is obtained by the use of ${}_a A$ and the contravariant metric $\overline{g} = g^{kl} \cdot \partial_k \cdot \partial_l$

$$\begin{aligned} A^{[j_1 \dots j_k]} &: = g^{j_1 \bar{i}_1} \cdot g^{j_2 \bar{i}_2} \cdot \dots \cdot g^{j_k \bar{i}_k} \cdot A_{[i_1 \dots i_k]} , \\ {}_a \overline{A} &= A^{[j_1 \dots j_k]} \cdot \partial_{j_1} \wedge \dots \wedge \partial_{j_k} = \\ &= g^{j_1 \bar{i}_1} \cdot g^{j_2 \bar{i}_2} \cdot \dots \cdot g^{j_k \bar{i}_k} \cdot A_{[i_1 \dots i_k]} \cdot \partial_{j_1} \wedge \dots \wedge \partial_{j_k} . \end{aligned} \quad (23)$$

Let $d\omega := \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_n} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_n}$ be the invariant volume element in M ($\dim M = n$), $d_g = \det(g_{ij}) < 0$. Let $d\omega$ acts on ${}_a\bar{A}$ as a mapping which maps ${}_a\bar{A}$ in a new tensor field $*({}_aA)$

$$\begin{aligned}
d\omega &: {}_a\bar{A} \rightarrow (d\omega)({}_a\bar{A}) := *({}_aA) \in \Lambda^{n-k}(M) , \\
*({}_aA) &= (d\omega)({}_a\bar{A}) := \frac{1}{k!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_{n-k} j_1 \dots j_k} \cdot A^{[\bar{j}_1 \dots \bar{j}_k]} \cdot dx^{i_1} \wedge \dots \\
&\dots \wedge dx^{i_{n-k}} = \\
&= \frac{1}{k!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_{n-k} j_1 \dots j_k} \cdot g^{\bar{j}_1 \bar{l}_1} \cdot g^{\bar{j}_2 \bar{l}_2} \cdot \dots \cdot g^{\bar{j}_k \bar{l}_k} \cdot A_{[l_1 \dots l_k]} \cdot dx^{i_1} \wedge \dots \\
&\dots \wedge dx^{i_{n-k}} = \\
&= *({}_aA)_{[i_1 \dots i_{n-k}]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_{n-k}} , \tag{24}
\end{aligned}$$

$$*({}_aA)_{[i_1 \dots i_{n-k}]} = \frac{1}{k!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_{n-k} j_1 \dots j_k} \cdot g^{\bar{j}_1 \bar{l}_1} \cdot g^{\bar{j}_2 \bar{l}_2} \cdot \dots \cdot g^{\bar{j}_k \bar{l}_k} \cdot A_{[l_1 \dots l_k]} . \tag{25}$$

Therefore, the Hodge (star) operator $*$ could be considered as defined by means of two mappings: $* = d\omega \circ \bar{\mathcal{G}}$:

(a) The mapping $\bar{\mathcal{G}} : {}_aA \rightarrow \bar{\mathcal{G}}({}_aA) := {}_a\bar{A}$, where ${}_aA \in {}^a\otimes_k(M) \equiv \Lambda^k(M)$, ${}_a\bar{A} \in {}^a\otimes^k(M)$. The operator $\bar{\mathcal{G}}$ is an operator, mapping a full covariant anti-symmetric tensor field into a full contravariant anti-symmetric tensor field.

(b) The mapping $d\omega : {}_a\bar{A} \rightarrow (d\omega)({}_a\bar{A}) = (d\omega)(\bar{\mathcal{G}}({}_aA)) = (d\omega \circ \bar{\mathcal{G}})({}_aA) := *({}_aA)$, where ${}_a\bar{A} \in {}^a\otimes^k(M)$, $*({}_aA) \in {}^a\otimes_{n-k}(M) \equiv \Lambda^{n-k}(M)$. The mapping $d\omega$ is an invariant n -form in M with $\dim M = n$, and at the same time, it is the invariant volume element in M . The form $d\omega$ acts on ${}_a\bar{A}$ by means of the contraction operator S over M applied k -times on ${}_a\bar{A}$.

2.2.2 Properties of the Hodge (star) operator

The main property of the star operator $*$ is related to its double action on a full covariant anti-symmetric tensor field. Let us now calculate the expression $*(*)({}_aA)$ by the use of the explicit definition of the star operator $*$.

$$\begin{aligned}
(({}_aA)) &= (d\omega)(\overline{*({}_aA)}) , \\
\overline{*({}_aA)} &= \overline{*({}_aA)}^{[i_1 \dots i_{n-k}]} \cdot \partial_{i_1} \wedge \dots \wedge \partial_{i_{n-k}} , \\
\overline{*({}_aA)}^{[i_1 \dots i_{n-k}]} &= g^{i_1 \bar{m}_1} \cdot \dots \cdot g^{i_{n-k} \bar{m}_{n-k}} \cdot (*({}_aA))_{[m_1 \dots m_{n-k}]} = \\
&= \frac{1}{k!} \cdot \sqrt{-d_g} \cdot \varepsilon_{m_1 \dots m_{n-k} j_1 \dots j_k} \cdot g^{i_1 \bar{m}_1} \cdot \dots \cdot g^{i_{n-k} \bar{m}_{n-k}} \cdot g^{\bar{j}_1 \bar{l}_1} \cdot g^{\bar{j}_2 \bar{l}_2} \cdot \dots \\
&\dots \cdot g^{\bar{j}_k \bar{l}_k} \cdot A_{[l_1 \dots l_k]} . \tag{26}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(({}_aA)) &= (d\omega)(\overline{*({}_aA)}) = \\
&= \frac{1}{(n-k)!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_k p_1 \dots p_{n-k}} (\overline{*({}_aA)})^{[p_1 \dots p_{n-k}]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k} = \\
&= \frac{1}{(n-k)!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \dots i_k p_1 \dots p_{n-k}} \cdot \\
&\cdot \frac{1}{k!} \cdot \sqrt{-d_g} \cdot \varepsilon_{m_1 \dots m_{n-k} j_1 \dots j_k} \cdot g^{\bar{p}_1 \bar{m}_1} \cdot \dots \cdot g^{\bar{p}_{n-k} \bar{m}_{n-k}} \cdot g^{\bar{j}_1 \bar{l}_1} \cdot g^{\bar{j}_2 \bar{l}_2} \cdot \dots \\
&\dots \cdot g^{\bar{j}_k \bar{l}_k} \cdot A_{[l_1 \dots l_k]} \cdot \\
&\cdot dx^{i_1} \wedge \dots \wedge dx^{i_k} =
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(n-k)!} \cdot \frac{1}{k!} \cdot (-d_g) \cdot \varepsilon_{i_1 \dots i_k p_1 \dots p_{n-k}} \cdot \\
&\quad \cdot \varepsilon_{m_1 \dots m_{n-k} j_1 \dots j_k} \cdot g^{\overline{m}_1 \overline{p}_1} \dots g^{\overline{m}_{n-k} \overline{p}_{n-k}} \cdot g^{\overline{j}_1 \overline{l}_1} \cdot g^{\overline{j}_2 \overline{l}_2} \dots g^{\overline{j}_k \overline{l}_k} \cdot A_{[l_1 \dots l_k]} \cdot \\
&\quad \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}
\end{aligned} \tag{27}$$

Since

$$\begin{aligned}
&\varepsilon_{m_1 \dots m_{n-k} j_1 \dots j_k} \cdot g^{\overline{m}_1 \overline{p}_1} \dots g^{\overline{m}_{n-k} \overline{p}_{n-k}} \cdot g^{\overline{j}_1 \overline{l}_1} \cdot g^{\overline{j}_2 \overline{l}_2} \dots g^{\overline{j}_k \overline{l}_k} = \\
&= \det(g^{\overline{i} \overline{j}}) \cdot \varepsilon^{p_1 \dots p_{n-k} l_1 \dots l_k},
\end{aligned} \tag{28}$$

we have

$$\begin{aligned}
(({}_a A)) &= \frac{1}{(n-k)!} \cdot \frac{1}{k!} \cdot (-d_g) \cdot \det(g^{\overline{i} \overline{j}}) \cdot \varepsilon_{i_1 \dots i_k p_1 \dots p_{n-k}} \cdot \varepsilon^{p_1 \dots p_{n-k} l_1 \dots l_k} \cdot A_{[l_1 \dots l_k]} \cdot \\
&\quad \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}.
\end{aligned} \tag{29}$$

On the other side,

$$\begin{aligned}
(-d_g) \cdot \det(g^{\overline{i} \overline{j}}) &= -\det(g_{kl}) \cdot (\det g^{\overline{i} \overline{j}}) = -\det(g_k^i) = -1, \\
\varepsilon_{i_1 \dots i_k p_1 \dots p_{n-k}} \cdot \varepsilon^{p_1 \dots p_{n-k} l_1 \dots l_k} &= (-1)^{k \cdot (n-k)} \cdot (n-k)! \cdot \varepsilon_{i_1 \dots i_k} \cdot \varepsilon^{l_1 \dots l_k} = \\
&= (-1)^{k \cdot (n-k)} \cdot (n-k)! \cdot a_{i_1 \dots i_k}^{l_1 \dots l_k}.
\end{aligned} \tag{30}$$

Then, it follows for $*(*({}_a A))$

$$\begin{aligned}
(({}_a A)) &= -\frac{1}{k! \cdot (n-k)!} \cdot (-1)^{k \cdot (n-k)} \cdot (n-k)! \cdot a_{i_1 \dots i_k}^{l_1 \dots l_k} \cdot A_{[l_1 \dots l_k]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k} = \\
&= -\frac{1}{k!} \cdot (-1)^{k \cdot (n-k)} \cdot a_{i_1 \dots i_k}^{l_1 \dots l_k} \cdot A_{[l_1 \dots l_k]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}.
\end{aligned} \tag{31}$$

Since

$$a_{i_1 \dots i_k}^{l_1 \dots l_k} \cdot A_{[l_1 \dots l_k]} = \frac{n!}{(n-k)!} \cdot A_{[i_1 \dots i_k]}, \tag{32}$$

we have as a result for $*(*({}_a A))$

$$\begin{aligned}
(({}_a A)) &= -\frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot A_{[i_1 \dots i_k]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k} = \\
&= -\frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot {}_a A.
\end{aligned} \tag{33}$$

If $\det(g_{ij}) = d_g > 0$ we have

$$\begin{aligned}
(({}_a A)) &= \frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot A_{[i_1 \dots i_k]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k} = \\
&= \frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot {}_a A.
\end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned}
(({}_a A)) &= \varepsilon \cdot \frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot {}_a A = (* \circ *)({}_a A), \\
* \circ * &= \varepsilon \cdot \frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot id,
\end{aligned} \tag{35}$$

with $\varepsilon = -1$ for $d_g < 0$ and $\varepsilon = 1$ for $d_g > 0$. The operator id is the identity operator. Since

$$\frac{n!}{k!} = \frac{k!(k+1)!}{k!} = (k+1)! , \quad (n-k)! = (n-1) \cdot (n-2) \cdot \dots \cdot (n-k) , \quad (36)$$

it follows that

$$\frac{n!}{(n-k)!k!} = \binom{n}{k} = C_n^k , \quad (37)$$

where C_n^k are binomial coefficients. Now, we can write as a final result for $* \circ *$

$$\begin{aligned} * \circ * &= \varepsilon \cdot \frac{n!}{(n-k)!k!} \cdot (-1)^{k \cdot (n-k)} \cdot id = * \circ * = \varepsilon \cdot C_n^k \cdot (-1)^{k \cdot (n-k)} \cdot id = \\ &= * \circ * = \varepsilon \cdot \binom{n}{k} \cdot (-1)^{k \cdot (n-k)} \cdot id . \end{aligned} \quad (38)$$

The Hodge (star) operator is a necessary tool if we further wish to find the relation between the vortex (rotation) velocity tensor and the vortex (rotation) vector.

2.3 Covariant divergency of a mixed tensor field

The operation of the covariant differentiation along a contravariant vector field can be extended to covariant differentiation along a contravariant tensor field.

The Lie derivative $\mathcal{L}_\xi u$ of a contravariant vector field u along a contravariant vector field ξ can be expressed by the use of the covariant differential operators ∇_ξ and ∇_u in the form

$$\mathcal{L}_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) , \quad \xi, u \in T(M) ,$$

where $T(\xi, u)$ is the contravariant torsion vector field

$$T(\xi, u) = T_{\alpha\beta}{}^\gamma \cdot \xi^\alpha \cdot u^\beta \cdot e_\gamma = T_{ij}{}^k \cdot \xi^i \cdot u^j \cdot \partial_k ,$$

constructed by means of the components $T_{\alpha\beta}{}^\gamma$ (or $T_{ij}{}^k$) of the contravariant torsion tensor field T .

The Lie derivative $\mathcal{L}_\xi V$ of a contravariant tensor field $V = V^A \cdot e_A = V^A \cdot \partial_A \in \otimes^m(M)$ along a contravariant vector field ξ can be written on the analogy of the relation for $\mathcal{L}_\xi u$ and by the use of the covariant differential operator ∇_ξ and an operator ∇_V in the form [30]

$$\mathcal{L}_\xi V = \nabla_\xi V - \nabla_V \xi - T(\xi, V) , \quad \xi \in T(M) , \quad V \in \otimes^m(M) ,$$

where

$$\begin{aligned} \nabla_V \xi &= -\xi^\alpha{}_{/\beta} \cdot S_{B\alpha}{}^{A\beta} \cdot V^B \cdot e_A = -\xi^i{}_{;j} \cdot S_{Ci}{}^{Aj} \cdot V^C \cdot \partial_A , \\ T(\xi, V) &= T_{B\gamma}{}^A \cdot \xi^\gamma \cdot V^B \cdot e_A = T_{Ck}{}^A \cdot \xi^k \cdot V^C \cdot \partial_A , \\ T_{B\gamma}{}^A &= T_{\beta\gamma}{}^\alpha \cdot S_{B\alpha}{}^{A\beta} , \quad T_{Ck}{}^A = T_{jk}{}^i \cdot S_{Ci}{}^{Aj} . \end{aligned}$$

$\nabla_V \xi$ appears as a definition of the action of the operator ∇_V on the vector field ξ . Let us now consider more closely this operator and its properties.

Let a mixed tensor field $K \in \otimes^k_l(M)$ be given in a non-co-ordinate (or co-ordinate) basis

$$\begin{aligned} K &= K^C{}_D \cdot e_C \otimes e^D = K^{C_1\alpha}{}_D \cdot e_{C_1} \otimes e_\alpha \otimes e^D , \\ e_C &= e_{C_1} \otimes e_\alpha , \quad e^D = e^{\alpha_1} \otimes \dots \otimes e^{\alpha_l} . \end{aligned}$$

The action of the operator ∇_V on the mixed tensor field K can be defined on the analogy of the action of ∇_V on a contravariant vector field ξ

$$\nabla_V K = -K^{C_1\alpha}{}_{D/\beta} \cdot S_{B\alpha}{}^{A\beta} \cdot V^B \cdot e_{C_1} \otimes e^D \otimes e_A, \quad (39)$$

where ∇_V is the covariant differential operator along a contravariant tensor field V

$$\nabla_V : K \Rightarrow \nabla_V K, \quad K \in \otimes^k_l(M), \quad V \in \otimes^m(M), \quad \nabla_V K \in \otimes^{k-1+m}_l(M).$$

Remark. There is an other possibility for a generalization of the action of ∇_V on a mixed tensor field

$$\nabla_V K = -\sum_{m=1}^k K^{\alpha_1 \dots \alpha_m \dots \alpha_k}{}_{D/\beta} \cdot S_{B\alpha_m}{}^{A\beta} \cdot V^B \cdot e_{\alpha_1} \otimes \dots \otimes e_{\alpha_{m-1}} \otimes e_{\alpha_{m+1}} \otimes \dots \otimes e_{\alpha_k} \otimes e^D \otimes e_A.$$

The result of the action of this operator on a contravariant vector field ξ is identical with the action of the above defined operator ∇_V .

Remark. A covariant differential operator along a contravariant tensor field can also be defined through its action on mixed tensor fields in the form

$$\overline{\nabla}_V K = K^C{}_{D/\beta} \cdot V^{A_1\beta} \cdot e_C \otimes e^D \otimes e_{A_1}.$$

The operator $\overline{\nabla}_V$ differs from ∇_V in its action on a contravariant vector field and does not appear as a generalization of the already defined operator by its action on a contravariant vector field. It appears as a new differential operator acting on mixed tensor fields.

The *covariant differential operator* ∇_V has the properties:

(a) Linear operator

$$\begin{aligned} \nabla_V(\alpha \cdot K_1 + \beta \cdot K_2) &= \alpha \cdot \nabla_V K_1 + \beta \cdot \nabla_V K_2, \\ \alpha, \beta &\in \mathbf{R} \text{ (or } \mathbf{C}), \quad K_1, K_2 \in \otimes^k_l(M). \end{aligned}$$

The proof of this property follows immediately from (39) and the linear property of the covariant differential operator along a basic contravariant vector field.

(b) Differential operator (not obeying the Leibniz rule)

$$\begin{aligned} \nabla_V(K \otimes S) &= \nabla_{e_\beta} K \otimes \overline{S}^\beta + K \otimes \nabla_V S, \\ K &= K^A{}_B \cdot e_A \otimes e^B, \quad \nabla_{e_\beta} K = K^A{}_{B/\beta} \cdot e_A \otimes e^B, \\ S &= \tilde{S}^C{}_D \cdot e_C \otimes e^D = \tilde{S}^{C_1\alpha}{}_D \cdot e_{C_1} \otimes e_\alpha \otimes e^D, \\ \overline{S}^\beta &= -\tilde{S}^{C_1\alpha}{}_D \cdot S_{E\alpha}{}^{F\beta} \cdot V^E \cdot e_{C_1} \otimes e^D \otimes e_F. \end{aligned}$$

The proof of this property follows from the action of the defined in (39) operator ∇_V and the properties of the covariant derivative of the product of the components of the tensor fields K and S .

If the tensor field V is given as a contravariant metric tensor field \overline{g} , then the covariant differential operator ∇_V ($V = \overline{g}$) will have additional properties connected with the properties of the contravariant metric tensor field.

Definition 1 *Contravariant metric differential operator $\nabla_{\overline{g}}$. Covariant differential operator ∇_V for $V = \overline{g}$.*

By means of the relations

$$\begin{aligned} -S_{B\alpha}{}^{A\beta} \cdot g^B \cdot e_A &= (g_\alpha^\sigma \cdot g^{\beta\kappa} + g_\alpha^\kappa \cdot g^{\beta\sigma}) \cdot e_\sigma \otimes e_\kappa = \\ &= (g_\alpha^\sigma \cdot g^{\beta\kappa} + g_\alpha^\kappa \cdot g^{\beta\sigma}) \cdot e_\sigma \cdot e_\kappa, \\ e_\sigma \cdot e_\kappa &= \frac{1}{2} \cdot (e_\sigma \otimes e_\kappa + e_\kappa \otimes e_\sigma), \end{aligned} \quad (40)$$

$$K^{C_1\alpha}{}_{D/\beta} \cdot g_\alpha^\sigma = K^{C_1\sigma}{}_{D/\beta} , \quad (41)$$

the action of the contravariant metric differential operator on a mixed tensor field K can be represented in the form

$$\begin{aligned} \nabla_{\bar{g}} K &= (K^{C_1\sigma}{}_{D/\beta} \cdot g^{\beta\kappa} + K^{C_1\kappa}{}_{D/\beta} \cdot g^{\beta\sigma}) \cdot e_{C_1} \otimes e^D \otimes e_\sigma \otimes e_\kappa = \\ &= (K^{C_1\sigma}{}_{D/\beta} \cdot g^{\beta\kappa} + K^{C_1\kappa}{}_{D/\beta} \cdot g^{\beta\sigma}) \cdot e_{C_1} \otimes e^D \otimes e_\sigma \cdot e_\kappa . \end{aligned} \quad (42)$$

The properties of the operator $\nabla_{\bar{g}}$ are determined additionally by the properties of the contravariant metric tensor field of second rank:

- (a) $\nabla_{\bar{g}} : K \Rightarrow \nabla_{\bar{g}} K$, $K \in \otimes^k{}_l(M)$, $\nabla_{\bar{g}} K \in \otimes^{k+1}{}_l(M)$.
- (b) Linear operator

$$\nabla_{\bar{g}}(\alpha \cdot K_1 + \beta \cdot K_2) = \alpha \cdot \nabla_{\bar{g}} K_1 + \beta \cdot \nabla_{\bar{g}} K_2 .$$

- (c) Differential operator (not obeying the Leibniz rule)

$$\begin{aligned} \nabla_{\bar{g}}(K \otimes S) &= \nabla_{e_\beta} K \otimes \bar{S}^\beta + K \otimes \nabla_{\bar{g}} S , \\ K &\in \otimes^k{}_l(M) , \quad S \in \otimes^m{}_r(M) , \\ K &= K^A{}_{B/\beta} \cdot e_A \otimes e^B , \quad \nabla_{e_\beta} K = K^A{}_{B/\beta} \cdot e_A \otimes e^B , \\ S &= \tilde{S}^C{}_D \cdot e_C \otimes e^D = \tilde{S}^{C_1\alpha}{}_D \cdot e_{C_1} \otimes e_\alpha \otimes e^D , \\ \bar{S}^\beta &= (\tilde{S}^{C_1\sigma}{}_D \cdot g^{\beta\kappa} + \tilde{S}^{C_1\kappa}{}_D \cdot g^{\beta\sigma}) \cdot e_{C_1} \otimes e^D \otimes e_\sigma \cdot e_\kappa . \end{aligned} \quad (43)$$

Remark. The definition of $\nabla_{\bar{g}}$ in (42) differs from the definition in [16], where $\nabla_{\bar{g}} \equiv \bar{\nabla}_{\bar{g}}$, i. e. the contravariant metric differential operator is defined in the last case as a special case of the covariant differential operator $\bar{\nabla}_V$ for $V = \bar{g}$.

The notion of covariant divergency of a mixed tensor field has been used in V_4 -spaces for the determination of conditions for the existence of local conserved quantities and in identities of the type of the first covariant Noether identity. Usually, the covariant divergency of a contravariant or mixed tensor field has been given in co-ordinate or non-co-ordinate basis in the form

$$\delta K = K^{A\beta}{}_{B/\beta} \cdot e_A \otimes e^B = K^{Ci}{}_{D;i} \cdot \partial_C \otimes dx^D , \quad (44)$$

where

$$K^{A\beta}{}_{B/\beta} = K^{A\beta}{}_{B/\gamma} \cdot g_\beta^\gamma , \quad K^{Ci}{}_{D;i} = K^{Ci}{}_{D;k} \cdot g_i^k . \quad (45)$$

For full anti-symmetric covariant tensor fields (differential forms) the covariant divergency (called also codifferential) δ is defined by means of the Hodge operator $*$, its reverse operator $*^{-1}$ and the external differential operator ${}_a\bar{d}$ in the form [?], [?] (pp. 147-149)

$$\delta = *^{-1} \circ {}_a\bar{d} \circ * . \quad (46)$$

Remark. The *Hodge operator* is constructed by means of the permutation (Levi-Chivita) symbols. It maps a full covariant anti-symmetric tensor of rank $(0, p) \equiv {}^a \otimes_p M \equiv \Lambda^p(M)$ in a full covariant anti-symmetric tensor of rank $(0, n-p) \equiv {}^a \otimes_{n-p} M \equiv \Lambda^{n-p}(M)$, where $\dim M = n$,

$$* : {}_a A \rightarrow {}_a A , \quad {}_a A \in \Lambda^p(M) , \quad * {}_a A \in \Lambda^{n-p}(M) ,$$

with

$$\begin{aligned} {}_a A &= A_{[i_1 \dots i_p]} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_p} , \quad * {}_a A = {}_a A_{[j_1 \dots j_{n-p}]} \cdot dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}} , \\ {}_a A_{[j_1 \dots j_{n-p}]} &= \frac{1}{p!} \cdot \varepsilon_{i_1 \dots i_p j_1 \dots j_{n-p}} \cdot A^{[i_1 \dots i_p]} , \quad A^{[i_1 \dots i_p]} = g^{i_1 \bar{k}_1} \dots g^{i_p \bar{k}_p} \cdot A_{[k_1 \dots k_p]} , \\ *^{-1} &= (-1)^{p \cdot (n-p)} \cdot * . \end{aligned}$$

By the use of the contravariant metric differential operator $\nabla_{\bar{g}}$, the covariant metric tensor field g and the contraction operator one can introduce the notion of covariant divergency of a mixed tensor field K with finite rank.

Definition 2 Covariant divergency δK of a mixed tensor field K

$$\delta K = \frac{1}{2} \cdot [\nabla_{\bar{g}} K]g = K^{A\beta}{}_{B/\beta} \cdot e_A \otimes e^B = K^{Ci}{}_{D;i} \cdot \partial_C \otimes dx^D ,$$

where

$$K = K^{A\beta}{}_{B/\beta} \cdot e_A \otimes e_\beta \otimes e^B = K^{Ci}{}_{D/\beta} \cdot \partial_C \otimes \partial_i \otimes dx^D , \\ K \in \otimes^k_l(M) , \quad k \geq 1 .$$

δ is called operator of the covariant divergency

$$\delta : K \Rightarrow \delta K , \quad K \in \otimes^k_l(M) , \quad \delta K \in \otimes^{k-1}_l(M) , \quad k \geq 1 .$$

Remark. The symbol δ has also been introduced for the variation operator. Both operators are different from each other and can easily be distinguished. Ambiguity would occur only if the symbol δ is used out of the context. In such a case, the definition of the symbol δ is necessary.

The properties of the operator of the covariant divergency δ are determined by the properties of the contravariant metric differential operator, the contraction operator and the metric tensor fields g and \bar{g}

(a) The operator of the covariant divergency δ is a linear operator

$$\delta(\alpha \cdot K_1 + \beta \cdot K_2) = \alpha \cdot \delta K_1 + \beta \cdot \delta K_2 , \\ \alpha, \beta \in \mathbf{R} \text{ (or } \mathbf{C} \text{)} , \quad K_1, K_2 \in \otimes^k_l(M) . \quad (47)$$

The proof of this property follows immediately from the definition of the covariant divergency.

(b) Action on a tensor product of tensor fields

$$\delta(K \otimes S) = \bar{\nabla}_S K + K \otimes \delta S , \quad (48)$$

where

$$K = K^A{}_{B/\beta} \cdot e_A \otimes e^B , \quad S = S^{C\beta}{}_{D/\beta} \cdot e_C \otimes e_\beta \otimes e^D , \\ \bar{\nabla}_S K = K^A{}_{B/\beta} \cdot S^{C\beta}{}_{D/\beta} \cdot e_A \otimes e^B \otimes e_C \otimes e^D \quad (\text{see above } \bar{\nabla}_V) . \quad (49)$$

The proof of this property follows from the properties of $\nabla_{\bar{g}}$ and from the relations

$$\frac{1}{2} [\nabla_{e_\beta} K \otimes \bar{S}^\beta]g = K^A{}_{B/\beta} \cdot S^{C\beta}{}_{D/\beta} \cdot e_A \otimes e^B \otimes e_C \otimes e^D , \quad (50)$$

$$\frac{1}{2} [K \otimes \nabla_{\bar{g}} S]g = K \otimes \frac{1}{2} \cdot [\nabla_{\bar{g}} S]g = K \otimes \delta S . \quad (51)$$

(c) Action on a contravariant vector field u

$$\delta u = \frac{1}{2} \cdot [\nabla_{\bar{g}} u]g = u^\beta{}_{/\beta} = u^i{}_{;i} . \quad (52)$$

(d) Action on the tensor product of two contravariant vector fields u and v

$$\delta(u \otimes v) = \nabla_v u + \delta u \cdot v , \quad \bar{\nabla}_v u = \nabla_v u . \quad (53)$$

(e) Action of the product of an invariant function L and a mixed tensor field K

$$\delta(L \cdot K) = \bar{\nabla}_K L + L \cdot \delta K , \quad (54)$$

$$\bar{\nabla}_K L = L_{/\beta} \cdot K^{A\beta}{}_{B/\beta} \cdot e_A \otimes e^B , \quad \delta K = K^{A\beta}{}_{B/\beta} \cdot e_A \otimes e^B , \\ L_{/\beta} = e_\beta L , \quad L_{;i} = L_{,i} , \\ K = K^{A\beta}{}_{B/\beta} \cdot e_A \otimes e_\beta \otimes e^B \in \otimes^k_l(M) . \quad (55)$$

Special case: Action of the product of an invariant function L and the contravariant metric tensor \bar{g} :

$$\delta(L \cdot \bar{g}) = (L_{/\beta} \cdot g^{\alpha\beta} + L \cdot g^{\alpha\beta}_{/\beta}) = (L_{;j} \cdot g^{ij} + L \cdot g^{ij}_{;j}) . \quad (56)$$

(f) Action on an anti-symmetric tensor product of two contravariant vector fields u and v

$$\begin{aligned} \delta(u \wedge v) &= \frac{1}{2} \cdot (\nabla_v u - \nabla_u v + \delta v \cdot u - \delta u \cdot v) = \\ &= -\frac{1}{2} \cdot [\mathcal{L}_u v + T(u, v) + \delta u \cdot v - \delta v \cdot u] . \end{aligned} \quad (57)$$

(g) Action on a full anti-symmetric contravariant tensor field A of second rank

$$\begin{aligned} \delta A &= \frac{1}{2} \cdot (A^{\alpha\beta} - A^{\beta\alpha})_{/\beta} \cdot e_\alpha = \frac{1}{2} \cdot (A^{ij} - A^{ji})_{;j} \cdot \partial_i , \\ A &= A^{\alpha\beta} \cdot e_\alpha \wedge e_\beta = A^{ij} \cdot \partial_i \wedge \partial_j , \quad A^{\alpha\beta} = -A^{\beta\alpha} . \end{aligned} \quad (58)$$

(h) Action on a tensor product of a contravariant vector field u , multiplied with an invariant function, and a covariant vector field $g(v)$ with the contravariant vector field v

$$\begin{aligned} \delta(\varepsilon \cdot u \otimes g(v)) &= (u\varepsilon) \cdot g(v) + \varepsilon \cdot [\delta u \cdot g(v) + (\nabla_u g)(v) + g(\nabla_u v)] = \\ &= [u\varepsilon + \varepsilon \cdot \delta u] \cdot g(v) + \varepsilon \cdot [(\nabla_u g)(v) + g(\nabla_u v)] , \\ \varepsilon &\in C^r(M) , \quad \varepsilon'(x^{k'}) = \varepsilon(x^k) , \quad u, v \in T(M) . \end{aligned} \quad (59)$$

Special case: $v \equiv u$:

$$\delta(\varepsilon \cdot u \otimes g(u)) = [u\varepsilon + \varepsilon \cdot \delta u] \cdot g(u) + \varepsilon \cdot [(\nabla_u g)(u) + g(a)] , \quad \nabla_u u = a . \quad (60)$$

Special case: $\varepsilon = 1$:

$$\delta(u \otimes g(v)) = \delta u \cdot g(v) + (\nabla_u g)(v) + g(\nabla_u v) . \quad (61)$$

2.4 Covariant divergency of a mixed tensor field of second rank

From the definition of the covariant divergency δK of a mixed tensor field K , the explicit form of the covariant divergency of tensor fields of second rank of the type 1 or 2 follows as

$$\delta G = \frac{1}{2} [\nabla_{\bar{g}} G] g = G_{\alpha}{}^{\beta}_{/\beta} \cdot e^{\alpha} = G_i{}^j{}_{;j} \cdot dx^i , \quad (62)$$

$$\delta \bar{G} = \frac{1}{2} [\nabla_{\bar{g}} \bar{G}] g = \bar{G}^{\beta}{}_{\alpha/\beta} \cdot e^{\alpha} = \bar{G}^j{}_{i;j} \cdot dx^i . \quad (63)$$

By the use of the relations (52) \div (57), (59) \div (61), and the expression [see (48) \div (51)]

$$\bar{\nabla}_v(g(u)) = \nabla_v(g(u)) = (\nabla_v g)(u) + g(\nabla_v u) , \quad (64)$$

$$\delta(g(u) \otimes v) = \delta v \cdot g(u) + (\nabla_v g)(u) + g(\nabla_v u) , \quad (65)$$

$$\delta((^G S)g) = (g_{\alpha\bar{\gamma}} \cdot ^G S^{\beta\gamma})_{/\beta} \cdot e^{\alpha} , \quad (66)$$

the covariant divergency of the representation of G by means the rest mass density ρ_G ($\varepsilon_G = \rho_G$)

$$G = \rho_G \cdot u \otimes g(u) + u \otimes g(^G \pi) + ^G s \otimes g(u) + (^G S)g ,$$

can be found in the form ($\nabla_u u = a$)

$$\begin{aligned} \delta G &= \rho_G \cdot g(a) + (u\rho_G + \rho_G \cdot \delta u + \delta^G s) \cdot g(u) + \delta u \cdot g(^G \pi) + g(\nabla_u ^G \pi) + \\ &+ g(\nabla_{^G s} u) + \rho_G \cdot (\nabla_u g)(u) + (\nabla_u g)(^G \pi) + (\nabla_{^G s} g)(u) + \delta((^G S)g) . \end{aligned} \quad (67)$$

$\bar{g}(\delta G)$ will have the form

$$\begin{aligned} \bar{g}(\delta G) = & \rho_G \cdot a + (u\rho_G + \rho_G \cdot \delta u + \delta^G s) \cdot u + \delta u \cdot {}^G \pi + \nabla_u {}^G \pi + \\ & + \nabla_{G_s} u + \rho_G \cdot \bar{g}(\nabla_u g)(u) + \bar{g}(\nabla_u g)({}^G \pi) + \bar{g}(\nabla_{G_s} g)(u) + \bar{g}(\delta({}^G S)g) . \end{aligned} \quad (68)$$

In a co-ordinate basis δG and $\bar{g}(\delta G)$ will have the forms

$$\begin{aligned} G_i{}^j{}_{;j} = & \rho_G \cdot a_i + (\rho_{G,j} \cdot u^j + \rho_G \cdot u^j{}_{;j} + {}^G s^j{}_{;j}) \cdot u_i + u^j{}_{;j} \cdot {}^G \pi_i + \\ & + g_{ij} \cdot ({}^G \pi^j{}_{;k} \cdot u^k + u^j{}_{;k} \cdot {}^G s^k) + g_{ij;k} \cdot (\rho_G \cdot u^k \cdot u^j + u^k \cdot {}^G \pi^j + {}^G s^k \cdot u^j) + \\ & + (g_{ik} \cdot {}^G S^{jk})_{;j} , \end{aligned} \quad (69)$$

$$\begin{aligned} a_i = & g_{ij} \cdot a^j , \quad a^i = u^i{}_{;j} \cdot u^j , \quad \rho_{G;i} = \rho_{G,i} , \quad u_i = g_{ik} \cdot u^k , \quad {}^G \pi_i = g_{ij} \cdot {}^G \pi^j , \\ g^{ik} \cdot G_k{}^j{}_{;j} = & \rho_G \cdot a^i + (\rho_{G,j} \cdot u^j + \rho_G \cdot u^j{}_{;j} + {}^G s^j{}_{;j}) \cdot u^i + u^j{}_{;j} \cdot {}^G \pi^i + \\ & + {}^G \pi^i{}_{;j} \cdot u^j + u^i{}_{;j} \cdot {}^G s^j + g^{il} \cdot g_{lj;k} \cdot (\rho_G \cdot u^k \cdot u^j + u^k \cdot {}^G \pi^j + {}^G s^k \cdot u^j) + \\ & + g^{il} \cdot (g_{lk} \cdot {}^G S^{jk})_{;j} . \end{aligned} \quad (70)$$

The relation between δG and $\delta \bar{G}$ follows from the relation between G and \bar{G}

$$\bar{G} = g(G)\bar{g} : \delta \bar{G} = \delta(g(G)\bar{g}) = \frac{1}{2}[\nabla_{\bar{g}}(g(G)\bar{g})] . \quad (71)$$

The *covariant divergency of the Kronecker tensor field* can be found in an analogous way as the covariant divergency of a tensor field of second rank of the type 1, since $Kr = g_\beta^\alpha \cdot e_\alpha \otimes e^\beta = g_\beta^i \cdot \partial_i \otimes dx^j$

$$\delta Kr = \frac{1}{2}[\nabla_{\bar{g}} Kr]g = g_\alpha^\beta{}_{/\beta} \cdot e^\alpha = g_{i;j}^j \cdot dx^i . \quad (72)$$

If we use the representation of Kr

$$Kr = \frac{1}{e} \cdot k \cdot u \otimes g(u) + u \otimes g({}^{Kr} \pi) + {}^{Kr} s \otimes g(u) + ({}^{Kr} S)g ,$$

then the covariant divergency δKr can be written in the form

$$\begin{aligned} \delta Kr = & \frac{1}{e} \cdot k \cdot g(a) + [u(\frac{1}{e} \cdot k) + \frac{1}{e} \cdot k \cdot \delta u + \delta {}^{Kr} s] \cdot g(u) + \\ & + \delta u \cdot g({}^{Kr} \pi) + g(\nabla_u {}^{Kr} \pi) + g(\nabla_{{}^{Kr} s} u) + \\ & + \frac{1}{e} \cdot k \cdot (\nabla_u g)(u) + (\nabla_u g)({}^{Kr} \pi) + (\nabla_{{}^{Kr} s} g)(u) + \delta({}^{Kr} S)g , \end{aligned} \quad (73)$$

or in the form

$$\begin{aligned} \bar{g}(\delta Kr) = & \frac{1}{e} \cdot k \cdot a + [u(\frac{1}{e} \cdot k) + \frac{1}{e} \cdot k \cdot \delta u + \delta {}^{Kr} s] \cdot u + \\ & + \delta u \cdot {}^{Kr} \pi + \nabla_u {}^{Kr} \pi + \nabla_{{}^{Kr} s} u + \\ & + \frac{1}{e} \cdot k \cdot \bar{g}(\nabla_u g)(u) + \bar{g}(\nabla_u g)({}^{Kr} \pi) + \bar{g}(\nabla_{{}^{Kr} s} g)(u) + \bar{g}(\delta({}^{Kr} S)g) . \end{aligned} \quad (74)$$

In a co-ordinate basis δKr and $\bar{g}(\delta Kr)$ will have the forms

$$\begin{aligned} g_i{}^j{}_{;j} = & \frac{1}{e} \cdot k \cdot a_i + [(\frac{1}{e} \cdot k)_{;j} \cdot u^j + \frac{1}{e} \cdot k \cdot u^j{}_{;j} + {}^{Kr} s^j{}_{;j}] \cdot u_i + \\ & + u^j{}_{;j} \cdot {}^{Kr} \pi_i + g_{ij} \cdot ({}^{Kr} \pi^j{}_{;k} \cdot u^k + u^j{}_{;k} \cdot {}^{Kr} s^k) + \\ & + g_{ij;k} \cdot (\frac{1}{e} \cdot k \cdot u^j \cdot u^k + {}^{Kr} \pi^j \cdot u^k + u^j \cdot {}^{Kr} s^k) + (g_{ik} \cdot {}^{Kr} S^{jk})_{;j} , \end{aligned} \quad (75)$$

$$\begin{aligned} g^{ik} \cdot g_k{}^j{}_{;j} = & \frac{1}{e} \cdot k \cdot a^i + [(\frac{1}{e} \cdot k)_{;j} \cdot u^j + \frac{1}{e} \cdot k \cdot u^j{}_{;j} + {}^{Kr} s^j{}_{;j}] \cdot u^i + \\ & + u^j{}_{;j} \cdot {}^{Kr} \pi^i + {}^{Kr} \pi^i{}_{;j} \cdot u^j + u^i{}_{;j} \cdot {}^{Kr} s^j + \\ & + g^{il} \cdot g_{lj;k} \cdot (\frac{1}{e} \cdot k \cdot u^j \cdot u^k + {}^{Kr} \pi^j \cdot u^k + u^j \cdot {}^{Kr} s^k) + g^{il} \cdot (g_{lk} \cdot {}^{Kr} S^{jk})_{;j} . \end{aligned} \quad (76)$$

3 Geometrical interpretation of the curvature tensor

The geometrical interpretation of the curvature tensor is well known for spaces with one affine connection [?] but all considerations, related to this topic are made in a given co-ordinate basis and not in an index-free manner. We will try now to find a reasonable geometrical interpretation of the curvature tensor in a covariant (index-free) manner using the introduced above mathematical tools.

Let us now consider a closed infinitesimal contour (quadrangle) $ACDBA$ build by a congruence of two parametric curves $x^i(\tau, \lambda)$. Let the tangent vectors to the curve $x^i(\tau, \lambda = \lambda_0 = \text{const.})$ and the curve $x^i(\tau = \tau_0 = \text{const.}, \lambda)$ be $u = d/d\tau$ and $d/d\lambda$ respectively.

3.1 Covariant transport of a vector on two different paths from a point to an other point of a closed infinitesimal contour

Let us now consider a covariant transport of two vectors from one to an other point of a closed infinitesimal contour $ACDBA$ on two different paths: from point A to point D across the point C and from point A to point D across the point B .

The point A has the co-ordinates $x^i(\tau_0, \lambda_0)$. The vectors u and ξ at the point A could be denoted as $u_{(\tau_0, \lambda_0)}$ and $\xi_{(\tau_0, \lambda_0)}$.

Then the vectors $\bar{u} = u_{(\tau_0+d\tau, \lambda_0)}$ and $\bar{\xi} = \xi_{(\tau_0+d\tau, \lambda_0)}$ at the point C [with co-ordinates $x^i(\tau_0 + d\tau, \lambda_0)$] could be related to the vectors $u_{(\tau_0, \lambda_0)}$ and $\xi_{(\tau_0, \lambda_0)}$ at the point A with co-ordinates $x^i(\tau_0, \lambda_0)$ by means of the relations

$$\underset{\text{from p. A}}{C} : \bar{u} = u_{(\tau_0+d\tau, \lambda_0)} = \left[(\exp[d\tau \cdot \frac{D}{d\tau}])u \right]_{(\tau_0, \lambda_0)}, \quad (77)$$

$$\underset{\text{from p. A}}{C} : \bar{\xi} = \xi_{(\tau_0+d\tau, \lambda_0)} = \left[(\exp[d\tau \cdot \frac{D}{d\tau}])\xi \right]_{(\tau_0, \lambda_0)}. \quad (78)$$

The vectors $\tilde{u} = u_{(\tau_0, \lambda_0+d\lambda)}$ and $\tilde{\xi} = \xi_{(\tau_0, \lambda_0+d\lambda)}$ at point B with co-ordinates $x^i(\tau_0, \lambda_0 + d\lambda)$ could be related to the vectors $u_{(\tau_0, \lambda_0)}$ and $\xi_{(\tau_0, \lambda_0)}$ by means of the relations

$$\underset{\text{from p. A}}{B} : \tilde{u} = u_{(\tau_0, \lambda_0+d\lambda)} = \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])u \right]_{(\tau_0, \lambda_0)}, \quad (79)$$

$$\underset{\text{from p. A}}{B} : \tilde{\xi} = \xi_{(\tau_0, \lambda_0+d\lambda)} = \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])\xi \right]_{(\tau_0, \lambda_0)}. \quad (80)$$

The vectors $\tilde{\tilde{u}} = u_{(\tau_0+d\tau, \lambda_0+d\lambda)}$ and $\tilde{\tilde{\xi}} = \xi_{(\tau_0+d\tau, \lambda_0+d\lambda)}$, obtained by the transport of the vectors $\tilde{u} = u_{(\tau_0, \lambda_0+d\lambda)}$ and $\tilde{\xi} = \xi_{(\tau_0, \lambda_0+d\lambda)}$ along the curve $x^i(\tau, \lambda_0 + d\lambda)$ from the point B with co-ordinates $x^i(\tau_0, \lambda_0 + d\lambda)$ to the point D with co-ordinates $x^i(\tau_0 + d\tau, \lambda_0 + d\lambda)$, could be related to the vectors $\tilde{u} = u_{(\tau_0, \lambda_0+d\lambda)}$ and $\tilde{\xi} = \xi_{(\tau_0, \lambda_0+d\lambda)}$ and then to the vectors $u_{(\tau_0, \lambda_0)}$ and $\xi_{(\tau_0, \lambda_0)}$ correspondingly by means of the relations

$$\begin{aligned} \underset{\text{from p. B}}{D} : \tilde{\tilde{u}} = u_{(\tau_0+d\tau, \lambda_0+d\lambda)} &= \left[(\exp[d\tau \cdot \frac{D}{d\tau}])u \right]_{(\tau_0, \lambda_0+d\lambda)} = \\ &= \left[(\exp[d\tau \cdot \frac{D}{d\tau}]) (\exp[d\lambda \cdot \frac{D}{d\lambda}])u \right]_{(\tau_0, \lambda_0)}, \end{aligned} \quad (81)$$

$$\begin{aligned}
\underset{\text{from } p. B}{D} : \quad \tilde{\xi} &= \xi_{(\tau_0+d\tau, \lambda_0+d\lambda)} = \left[(\exp[d\tau \cdot \frac{D}{d\tau}])\xi \right]_{(\tau_0, \lambda_0+d\lambda)} = \\
&= \left[(\exp[d\tau \cdot \frac{D}{d\tau}])(\exp[d\lambda \cdot \frac{D}{d\lambda}])\xi \right]_{(\tau_0, \lambda_0)} . \tag{82}
\end{aligned}$$

Until now we have considered the transport of the vectors u and ξ from point A to point C and from the point A to the point D across the point B .

On the other side, the vectors $\bar{\bar{u}} = u_{(\tau_0+d\tau, \lambda_0+d\lambda)}$ and $\bar{\bar{\xi}} = \xi_{(\tau_0+d\tau, \lambda_0+d\lambda)}$, obtained by the transport of the vectors $\bar{u} = u_{(\tau_0+d\tau, \lambda_0)}$ and $\bar{\xi} = \xi_{(\tau_0+d\tau, \lambda_0)}$ along the curve $x^i(\tau_0 + d\tau, \lambda)$ from point C with co-ordinates $x^i(\tau_0 + d\tau, \lambda_0)$ to the point D with co-ordinates $x^i(\tau_0 + d\tau, \lambda_0 + d\lambda)$, could be related to the vectors $\bar{u} = u_{(\tau_0+d\tau, \lambda_0)}$ and $\bar{\xi} = \xi_{(\tau_0+d\tau, \lambda_0)}$ and then to the vectors $u_{(\tau_0, \lambda_0)}$ and $\xi_{(\tau_0, \lambda_0)}$ correspondingly by means of the relations

$$\begin{aligned}
\underset{\text{from } p. C}{D} : \quad \bar{\bar{u}} &= u_{(\tau_0+d\tau, \lambda_0+d\lambda)} = \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])u \right]_{(\tau_0+d\tau, \lambda_0)} = \\
&= \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])(\exp[d\tau \cdot \frac{D}{d\tau}])u \right]_{(\tau_0, \lambda_0)} , \tag{83}
\end{aligned}$$

$$\begin{aligned}
\underset{\text{from } p. C}{D} : \quad \bar{\bar{\xi}} &= \xi_{(\tau_0+d\tau, \lambda_0+d\lambda)} = \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])\xi \right]_{(\tau_0+d\tau, \lambda_0)} = \\
&= \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])(\exp[d\tau \cdot \frac{D}{d\tau}])\xi \right]_{(\tau_0, \lambda_0)} . \tag{84}
\end{aligned}$$

For an other contravariant vector field v we have analogous relations as for the vector fields u and ξ if the vector v is transported along the infinitesimal contour $ACDBA$ from point A to point D across point C or from point A to point D across point B

$$\begin{aligned}
\underset{\text{from } p. B}{D} : \quad \tilde{v} &= v_{(\tau_0+d\tau, \lambda_0+d\lambda)} = \left[(\exp[d\tau \cdot \frac{D}{d\tau}])v \right]_{(\tau_0, \lambda_0+d\lambda)} = \\
&= \left[(\exp[d\tau \cdot \frac{D}{d\tau}])(\exp[d\lambda \cdot \frac{D}{d\lambda}])v \right]_{(\tau_0, \lambda_0)} , \tag{85}
\end{aligned}$$

$$\begin{aligned}
\underset{\text{from } p. C}{D} : \quad \bar{\bar{v}} &= v_{(\tau_0+d\tau, \lambda_0+d\lambda)} = \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])v \right]_{(\tau_0+d\tau, \lambda_0)} = \\
&= \left[(\exp[d\lambda \cdot \frac{D}{d\lambda}])(\exp[d\tau \cdot \frac{D}{d\tau}])v \right]_{(\tau_0, \lambda_0)} . \tag{86}
\end{aligned}$$

Now, we can compare the vectors \tilde{u} and $\bar{\bar{u}}$ as well as $\tilde{\xi}$ and $\bar{\bar{\xi}}$ with respect to the vectors u and ξ correspondingly. For $d\tau = d\lambda = \varepsilon \ll 1$, we have

$$\bar{\bar{u}} = \left[(\exp[\varepsilon \cdot \frac{D}{d\lambda}])(\exp[\varepsilon \cdot \frac{D}{d\tau}])u \right]_{(\tau_0, \lambda_0)} , \tag{87}$$

$$\bar{\bar{\xi}} = \left[(\exp[\varepsilon \cdot \frac{D}{d\lambda}])(\exp[\varepsilon \cdot \frac{D}{d\tau}])\xi \right]_{(\tau_0, \lambda_0)} , \tag{88}$$

$$\bar{\bar{v}} = \left[(\exp[\varepsilon \cdot \frac{D}{d\lambda}])(\exp[\varepsilon \cdot \frac{D}{d\tau}])v \right]_{(\tau_0, \lambda_0)} , \tag{89}$$

$$\tilde{\tilde{u}} = \left[(\exp[\varepsilon \cdot \frac{D}{d\tau}]) (\exp[\varepsilon \cdot \frac{D}{d\lambda}]) u \right]_{(\tau_0, \lambda_0)} , \quad (90)$$

$$\tilde{\tilde{\xi}} = \left[(\exp[\varepsilon \cdot \frac{D}{d\tau}]) (\exp[\varepsilon \cdot \frac{D}{d\lambda}]) \xi \right]_{(\tau_0, \lambda_0)} , \quad (91)$$

$$\tilde{\tilde{v}} = \left[(\exp[\varepsilon \cdot \frac{D}{d\tau}]) (\exp[\varepsilon \cdot \frac{D}{d\lambda}]) v \right]_{(\tau_0, \lambda_0)} . \quad (92)$$

Up to the second order, we obtain for the vectors $\bar{\bar{u}}$, $\bar{\bar{\xi}}$, and $\bar{\bar{v}}$ and for the vectors $\tilde{\tilde{u}}$, $\tilde{\tilde{\xi}}$, and $\tilde{\tilde{v}}$

$$\begin{aligned} \bar{\bar{u}} = & u_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left(\frac{Du}{d\tau} + \frac{Du}{d\lambda} \right)_{(\tau_0, \lambda_0)} + \varepsilon^2 \cdot \left[\left(\frac{D}{d\lambda} \circ \frac{D}{d\tau} \right) u \right]_{(\tau_0, \lambda_0)} + \\ & + \frac{1}{2} \cdot \varepsilon^2 \cdot \left(\frac{D^2 u}{d\lambda^2} + \frac{D^2 u}{d\tau^2} \right)_{(\tau_0, \lambda_0)} + O(\varepsilon^3) , \end{aligned} \quad (93)$$

$$\begin{aligned} \tilde{\tilde{u}} = & u_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left(\frac{Du}{d\lambda} + \frac{Du}{d\tau} \right)_{(\tau_0, \lambda_0)} + \varepsilon^2 \cdot \left[\left(\frac{D}{d\tau} \circ \frac{D}{d\lambda} \right) u \right]_{(\tau_0, \lambda_0)} + \\ & + \frac{1}{2} \cdot \varepsilon^2 \cdot \left(\frac{D^2 u}{d\lambda^2} + \frac{D^2 u}{d\tau^2} \right)_{(\tau_0, \lambda_0)} + O(\varepsilon^3) . \end{aligned} \quad (94)$$

The difference between the vectors $\bar{\bar{u}}$ and $\tilde{\tilde{u}}$ (transported along different paths from point A to point D) can be found up to the second order of ε in the form

$$\bar{\bar{u}} - \tilde{\tilde{u}} = \varepsilon^2 \cdot \left[\left(\frac{D}{d\lambda} \circ \frac{D}{d\tau} - \frac{D}{d\tau} \circ \frac{D}{d\lambda} \right) u \right]_{(\tau_0, \lambda_0)} + O(\varepsilon^3) , \quad (95)$$

or in the form

$$\tilde{\tilde{u}} - \bar{\bar{u}} = \varepsilon^2 \cdot \left[\left(\frac{D}{d\tau} \circ \frac{D}{d\lambda} - \frac{D}{d\lambda} \circ \frac{D}{d\tau} \right) u \right]_{(\tau_0, \lambda_0)} + O(\varepsilon^3) . \quad (96)$$

The last expression could also be written in the form (having in mind that $D/d\tau = \nabla_u$ and $D/d\lambda = \nabla_\xi$)

$$\begin{aligned} \tilde{\tilde{u}} - \bar{\bar{u}} &= \varepsilon^2 \cdot [(\nabla_u \nabla_\xi - \nabla_\xi \nabla_u) u]_{(\tau_0, \lambda_0)} + O(\varepsilon^3) , \\ \bar{\bar{u}} - \tilde{\tilde{u}} &= \varepsilon^2 \cdot [(\nabla_\xi \nabla_u - \nabla_u \nabla_\xi) u]_{(\tau_0, \lambda_0)} + O(\varepsilon^3) . \end{aligned} \quad (97)$$

Since $\nabla_\xi \nabla_u - \nabla_u \nabla_\xi = R(\xi, u) + \nabla_{\mathcal{L}_\xi u}$, we obtain

$$\bar{\bar{u}} - \tilde{\tilde{u}} = \varepsilon^2 \cdot \{ [R(\xi, u)] u + \nabla_{\mathcal{L}_\xi u} u \}_{(\tau_0, \lambda_0)} + O(\varepsilon^3) . \quad (98)$$

For a given vector field $v \in T(M)$ we obtain in an analogous relation

$$\bar{\bar{v}} - \tilde{\tilde{v}} = \varepsilon^2 \cdot \{ [R(\xi, u)] v + \nabla_{\mathcal{L}_\xi u} v \}_{(\tau_0, \lambda_0)} + O(\varepsilon^3) . \quad (99)$$

Therefore, the curvature operator acting on a contravariant vector field v determines the difference between the vectors, obtained by the transport of the vector v along two different paths constructing an infinitesimal closed contour. If the curves $x^i(\tau, \lambda_0)$ and $x^i(\tau_0, \lambda)$ are co-ordinate lines then $\mathcal{L}_\xi u = 0$ and therefore, for an infinitesimal closed co-ordinate contour we have

$$\bar{\bar{v}} - \tilde{\tilde{v}} = \varepsilon^2 \cdot \{ [R(\xi, u)] v \}_{(\tau_0, \lambda_0)} + O(\varepsilon^3) , \quad \mathcal{L}_\xi u = 0 . \quad (100)$$

We have obtained (in a covariant manner) a well known result leading to the geometrical interpretation of the curvature tensor in $[R(\xi, u)]v$ as a measure for the difference between the results of the different transports (on different paths) of one and the same vector from one to an other point of a closed contour.

3.2 Covariant transport of a vector along a closed infinitesimal contour from one and to the same point of the contour

Let us now consider the transport of a contravariant vector v from point A along the closed infinitesimal contour $ACDBA$ to the same point A .

At the point A we the vector v could be denoted as $v_{(\tau_0, \lambda_0)}$. At point B the transported vector $v_{(\tau_0, \lambda_0 + d\lambda)}$ could be represented by means of the vector $v_{(\tau_0, \lambda_0)}$ as

$${}_{from \ p. \ A}^B : \tilde{v} = v_{(\tau_0, \lambda_0 + d\lambda)} = \left[\exp\left[d\lambda \cdot \frac{D}{d\lambda}\right] v \right]_{(\tau_0, \lambda_0)} . \quad (101)$$

At the point D , the transported from point B vector $v_{(\tau_0, \lambda_0 + d\lambda)}$ could be represented in the form

$$\begin{aligned} {}_{from \ p. \ B}^D : \tilde{\tilde{v}} = v_{(\tau_0 + d\tau, \lambda_0 + d\lambda)} &= \left[\exp\left[d\tau \cdot \frac{D}{d\tau}\right] v \right]_{(\tau_0, \lambda_0 + d\lambda)} = \\ &= \left[\exp\left[d\tau \cdot \frac{D}{d\tau}\right] \left(\exp\left[d\lambda \cdot \frac{D}{d\lambda}\right] v \right) \right]_{(\tau_0, \lambda_0)} . \end{aligned} \quad (102)$$

At point C , the transported from point D vector $v_{(\tau_0 + d\tau, \lambda_0 + d\lambda)}$ can be written in the forms

$$\begin{aligned} {}_{from \ p. \ D}^C : v_{ABDC} = v_{(\tau_0 + d\tau, \lambda_0)} &= \left[\exp\left[-d\lambda \cdot \frac{D}{d\lambda}\right] v \right]_{(\tau_0 + d\tau, \lambda_0 + d\lambda)} = \\ &= \left[\exp\left[-d\lambda \cdot \frac{D}{d\lambda}\right] \left(\exp\left[d\tau \cdot \frac{D}{d\tau}\right] v \right) \right]_{(\tau_0, \lambda_0 + d\lambda)} \\ &= \left[\exp\left[-d\lambda \cdot \frac{D}{d\lambda}\right] \left(\exp\left[d\tau \cdot \frac{D}{d\tau}\right] \left(\exp\left[d\lambda \cdot \frac{D}{d\lambda}\right] v \right) \right) \right]_{(\tau_0, \lambda_0)} . \end{aligned} \quad (103)$$

At point A , the transported from point C vector $v_{(\tau_0 + d\tau, \lambda_0)}$ can be represented in the forms

$$\begin{aligned} {}_{from \ p. \ C}^A : v_{ABDCA} &= \left[\exp\left[-d\tau \cdot \frac{D}{d\tau}\right] v \right]_{(\tau_0 + d\tau, \lambda_0)} = \\ &= \left[\exp\left[-d\tau \cdot \frac{D}{d\tau}\right] \left(\exp\left[-d\lambda \cdot \frac{D}{d\lambda}\right] \left(\exp\left[d\tau \cdot \frac{D}{d\tau}\right] \left(\exp\left[d\lambda \cdot \frac{D}{d\lambda}\right] v \right) \right) \right) \right]_{(\tau_0, \lambda_0)} \end{aligned} \quad (104)$$

If we use the explicit form of the exponent of the covariant differential with $d\tau = d\lambda = \varepsilon$, we can find the expression for v_{ABDCA} in the form

$$\begin{aligned} v_{ABDCA} &= \left\{ \left[1 - \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \dots \right] \circ \left[1 - \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right] \circ \right. \\ &\quad \left. \circ \left[1 + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \dots \right] \circ \left[1 + \right. \right. \\ &\quad \left. \left. + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right] v \right\}_{(\tau_0, \lambda_0)} . \end{aligned} \quad (105)$$

Up to the second order of ε , we obtain the expressions

$$\begin{aligned} &\left\{ \left(\left(1 - \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} - \varepsilon \cdot \frac{D}{d\lambda} + \varepsilon^2 \cdot \frac{D}{d\tau} \circ \frac{D}{d\lambda} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + O(\varepsilon^3) \right) \circ \right. \right. \\ &\quad \left. \left. \circ \left(1 + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \varepsilon \cdot \frac{D}{d\lambda} + \varepsilon^2 \cdot \frac{D}{d\tau} \circ \frac{D}{d\lambda} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + O(\varepsilon^3) \right) \right) v \right\}_{(\tau_0, \lambda_0)} = \end{aligned}$$

$$\begin{aligned}
&= \{ [1 - \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} - \varepsilon \cdot \frac{D}{d\lambda} + \varepsilon^2 \cdot \frac{D}{d\tau} \circ \frac{D}{d\lambda} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \\
&\quad + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \varepsilon \cdot \frac{D}{d\lambda} + \varepsilon^2 \cdot \frac{D}{d\tau} \circ \frac{D}{d\lambda} + \frac{1}{2} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} - \\
&\quad - \varepsilon^2 \cdot \frac{D^2}{d\tau^2} - \varepsilon^2 \cdot \frac{D}{d\lambda} \circ \frac{D}{d\tau} - \varepsilon^2 \cdot \frac{D}{d\tau} \circ \frac{D}{d\lambda} - \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + O(\varepsilon^3)] v \}_{(\tau_0, \lambda_0)} \\
v_{ABDCA} &= \{ [1 + \varepsilon^2 \cdot \frac{D}{d\tau} \circ \frac{D}{d\lambda} - \varepsilon^2 \cdot \frac{D}{d\lambda} \circ \frac{D}{d\tau} + O(\varepsilon^3)] v \}_{(\tau_0, \lambda_0)} , \quad (106)
\end{aligned}$$

$$v_{ABDCA} \approx v_{(\tau_0, \lambda_0)} + \varepsilon^2 \cdot \left\{ \left[\frac{D}{d\tau} \circ \frac{D}{d\lambda} - \frac{D}{d\lambda} \circ \frac{D}{d\tau} \right] v \right\}_{(\tau_0, \lambda_0)} . \quad (107)$$

The difference between the vector $v_{(\tau_0, \lambda_0)}$ and the vector v_{ABDCA} , transported along the contour $ABDCA$ from the point A to the same point A , can be found up to the second order of ε as

$$v_{ABDCA} - v_{(\tau_0, \lambda_0)} \approx \varepsilon^2 \cdot \left\{ \left[\frac{D}{d\tau} \circ \frac{D}{d\lambda} - \frac{D}{d\lambda} \circ \frac{D}{d\tau} \right] v \right\}_{(\tau_0, \lambda_0)} . \quad (108)$$

Since

$$\begin{aligned}
\frac{D}{d\tau} &= \nabla_u \quad , \quad u = \frac{d}{d\tau} \quad , \quad \frac{D}{d\lambda} = \nabla_\xi \quad , \quad \xi = \frac{d}{d\lambda} \quad , \\
\left[\frac{D}{d\tau} \circ \frac{D}{d\lambda} - \frac{D}{d\lambda} \circ \frac{D}{d\tau} \right] v &= [\nabla_u \nabla_\xi - \nabla_\xi \nabla_u] v = [R(u, \xi) + \nabla_{\mathcal{L}_u \xi}] v = \\
&= [R(u, \xi)] v + \nabla_{\mathcal{L}_u \xi} v = -[R(\xi, u)] v - \nabla_{\mathcal{L}_\xi u} v , \quad (109)
\end{aligned}$$

it follows that

$$v_{ABDCA} \approx v_{(\tau_0, \lambda_0)} - \varepsilon^2 \cdot \{ [R(\xi, u)] v - \nabla_{\mathcal{L}_\xi u} v \}_{(\tau_0, \lambda_0)} , \quad (110)$$

$$v_{(\tau_0, \lambda_0)} - v_{ABDCA} \approx \varepsilon^2 \cdot \{ [R(\xi, u)] v - \nabla_{\mathcal{L}_\xi u} v \}_{(\tau_0, \lambda_0)} . \quad (111)$$

Therefore, up to the second order of ε we have

$$\begin{aligned}
v_{(\tau_0, \lambda_0)} - v_{ABDCA} &= \bar{\bar{v}} - \tilde{\tilde{v}} = v_{ACD} - v_{ABD} , \quad (112) \\
v_{ACD} &= \bar{\bar{v}} \quad , \quad v_{ABD} = \tilde{\tilde{v}} .
\end{aligned}$$

The curvature tensor in $[R(\xi, u)]v$ could also be interpreted as a measure for the deviation of a vector, transported along a infinitesimal closed contour, from the vector remaining at the same point from which the transport began.

In an analogous way, we obtain for a covariant vector field $p \in T^*(M)$ the relation

$$p_{(\tau_0, \lambda_0)} - p_{ABDCA} \approx \varepsilon^2 \cdot \{ [R(\xi, u)] p - \nabla_{\mathcal{L}_\xi u} p \}_{(\tau_0, \lambda_0)} . \quad (113)$$

3.3 Action of the contraction operator S on the pairs $(v, p)_{(\tau_0, \lambda_0)}$ and (v_{ABDCA}, p_{ABDCA})

From the relations

$$v_{(\tau_0, \lambda_0)} \approx v_{ABDCA} + \varepsilon^2 \cdot \{ [R(\xi, u)] v - \nabla_{\mathcal{L}_\xi u} v \}_{(\tau_0, \lambda_0)} , \quad (114)$$

$$p_{(\tau_0, \lambda_0)} \approx p_{ABDCA} + \varepsilon^2 \cdot \{ [R(\xi, u)] p - \nabla_{\mathcal{L}_\xi u} p \}_{(\tau_0, \lambda_0)} , \quad (115)$$

it follows the result of the action of the contraction operator S in the form

$$\begin{aligned}
S(v, p)_{(\tau_0, \lambda_0)} &= S \left(\begin{array}{c} v_{ABDCA} + \varepsilon^2 \cdot \{[R(\xi, u)]v - \nabla_{\mathcal{L}_\xi u} v\}_{(\tau_0, \lambda_0)}, \\ p_{ABDCA} + \varepsilon^2 \cdot \{[R(\xi, u)]p - \nabla_{\mathcal{L}_\xi u} p\}_{(\tau_0, \lambda_0)} \end{array} \right) = \\
&= S(v_{ABDCA}, p_{ABDCA}) + \\
&\quad + \varepsilon^2 \cdot S(\{[R(\xi, u)]v - \nabla_{\mathcal{L}_\xi u} v\}_{(\tau_0, \lambda_0)}, p_{ABDCA}) + \\
&\quad + \varepsilon^2 \cdot S(v_{ABDCA}, \{[R(\xi, u)]p - \nabla_{\mathcal{L}_\xi u} p\}_{(\tau_0, \lambda_0)}) + \\
&\quad + \varepsilon^4 \cdot S(\{[R(\xi, u)]v - \nabla_{\mathcal{L}_\xi u} v\}_{(\tau_0, \lambda_0)}, \{[R(\xi, u)]p - \nabla_{\mathcal{L}_\xi u} p\}_{(\tau_0, \lambda_0)}) \quad (116)
\end{aligned}$$

For an infinitesimal closed contour, constructed from co-ordinate lines with tangent vectors u and ξ , the relation $\mathcal{L}_\xi u = -\mathcal{L}_u \xi = 0$ is valid and the above relations up to the second order of ε have a simpler form

$$\begin{aligned}
S(v, p)_{(\tau_0, \lambda_0)} &\approx S(v_{ABDCA} + \varepsilon^2 \cdot \{[R(\xi, u)]v\}_{(\tau_0, \lambda_0)}, p_{ABDCA} + \varepsilon^2 \cdot \{[R(\xi, u)]p\}_{(\tau_0, \lambda_0)}) = \\
&\approx S(v_{ABDCA}, p_{ABDCA}) + \varepsilon^2 \cdot S(\{[R(\xi, u)]v\}_{(\tau_0, \lambda_0)}, p_{ABDCA}) + \\
&\quad + \varepsilon^2 \cdot S(v_{ABDCA}, \{[R(\xi, u)]p\}_{(\tau_0, \lambda_0)}) \quad , \quad (117) \\
S(v_{ABDCA}, p_{ABDCA}) &\approx S(v, p)_{(\tau_0, \lambda_0)} - \\
&\quad - \varepsilon^2 \cdot \{S(\{[R(\xi, u)]v\}_{(\tau_0, \lambda_0)}, p_{ABDCA}) + S(v_{ABDCA}, \{[R(\xi, u)]p\}_{(\tau_0, \lambda_0)})\} \quad . \quad (118)
\end{aligned}$$

On the other side, because of the relations

$$v_{ABDCA} \approx v_{(\tau_0, \lambda_0)} - \varepsilon^2 \cdot \{[R(\xi, u)]v\}_{(\tau_0, \lambda_0)} \quad , \quad (119)$$

$$p_{ABDCA} \approx p_{(\tau_0, \lambda_0)} - \varepsilon^2 \cdot \{[R(\xi, u)]p\}_{(\tau_0, \lambda_0)} \quad , \quad (120)$$

we obtain for $S(v_{ABDCA}, p_{ABDCA})$ up to the second order of ε

$$\begin{aligned}
S(v_{ABDCA}, p_{ABDCA}) &\approx S(v, p)_{(\tau_0, \lambda_0)} - \\
&\quad - \varepsilon^2 \cdot \{S([R(\xi, u)]v, p) + S(v, [R(\xi, u)]p)\}_{(\tau_0, \lambda_0)} \quad (121)
\end{aligned}$$

In a co-ordinate basis, the contravariant vector $[R(\xi, u)]v$ and the covariant vector $[R(\xi, u)]p$ could be written in the forms [13]

$$[R(\xi, u)]v = R^l_{kij} \cdot \xi^i \cdot u^j \cdot v^k \cdot \partial_l \quad , \quad (122)$$

$$[R(\xi, u)]p = p_k \cdot P^k_{mij} \cdot \xi^i \cdot u^j \cdot dx^m \quad . \quad (123)$$

By the use of the last expressions for $[R(\xi, u)]v$ and $[R(\xi, u)]p$ in the expressions for $S(v_{ABDCA}, p_{ABDCA})$, it follows that

$$\begin{aligned}
S(v_{ABDCA}, p_{ABDCA}) &\approx S(v, p)_{(\tau_0, \lambda_0)} - \\
&\quad - \varepsilon^2 \cdot [R^l_{kij} \cdot \xi^i \cdot u^j \cdot v^k \cdot S(\partial_l, p) + \\
&\quad + p_k \cdot P^k_{mij} \cdot \xi^i \cdot u^j \cdot S(v, dx^m)]_{(\tau_0, \lambda_0)} \quad . \quad (124)
\end{aligned}$$

Since

$$S(\partial_l, p) = S(\partial_l, p_m \cdot dx^m) = p_m \cdot S(\partial_l, dx^m) = p_m \cdot f^m_l \quad , \quad (125)$$

$$S(v, dx^m) = S(v^n \cdot \partial_n, dx^m) = v^n \cdot S(\partial_n, dx^m) = v^n \cdot f^m_n \quad , \quad (126)$$

we can find the relations

$$\begin{aligned}
S(v_{ABDCA}, p_{ABDCA}) &\approx S(v, p)_{(\tau_0, \lambda_0)} - \\
&\quad - \varepsilon^2 \cdot [R^l_{kij} \cdot f^m_l \cdot \xi^i \cdot u^j \cdot v^k \cdot p_m + \\
&\quad + P^k_{mij} \cdot f^m_n \cdot \xi^i \cdot u^j \cdot v^n \cdot p_k]_{(\tau_0, \lambda_0)} \quad , \quad (127)
\end{aligned}$$

$$\begin{aligned}
S(v_{ABDCA}, p_{ABDCA}) &\approx S(v, p)_{(\tau_0, \lambda_0)} - \\
&\quad - \varepsilon^2 \cdot [R^{\overline{m}}_{kij} \cdot \xi^i \cdot u^j \cdot v^k \cdot p_m + \\
&\quad + P^k_{\overline{n}ij} \cdot \xi^i \cdot u^j \cdot v^n \cdot p_k]_{(\tau_0, \lambda_0)} \quad , \quad (128)
\end{aligned}$$

$$\begin{aligned}
S(v_{ABDCA}, p_{ABDCA}) &\approx S(v, p)_{(\tau_0, \lambda_0)} - \\
&\quad - \varepsilon^2 \cdot [(R^{\overline{k}}_{lij} + P^k_{\overline{l}ij}) \cdot \xi^i \cdot u^j \cdot v^l \cdot p_k]_{(\tau_0, \lambda_0)} \quad (129)
\end{aligned}$$

On the other side, we have the integrability condition for the existence of the contraction operator S [13]

$$R^{\overline{k}}_{lij} + P^k_{\overline{l}ij} = 0 \quad . \quad (130)$$

Because of the integrability condition, the last term of the right of the expression for $S(v_{ABDCA}, p_{ABDCA})$ is equal to zero and it follows that

$$S(v_{ABDCA}, p_{ABDCA}) \approx S(v, p)_{(\tau_0, \lambda_0)} \quad . \quad (131)$$

This means that the result of the action of the contraction operator S on a contravariant and covariant vectors does not change after the transport of both the vectors along an infinitesimal closed co-ordinate contour (quadrangle).

In an analogous way, for the pairs $(\overline{v}, \overline{p})$ and $(\widetilde{v}, \widetilde{p})$, transported on different paths from point A to point D of an infinitesimal closed co-ordinate contour $ABDCA$, we obtain the relation

$$S(\overline{v}, \overline{p})_{(\tau_0+d\tau, \lambda_0+d\lambda)} = S(\widetilde{v}, \widetilde{p})_{(\tau_0+d\tau, \lambda_0+d\lambda)} \quad , \quad d\tau = d\lambda = \varepsilon \ll 1 \quad . \quad (132)$$

This means that the result of the action of the contraction operator S on a contravariant and a covariant tensors does not depend on the path on which theses tensors are transported from one to an other point of a closed infinitesimal co-ordinate contour.

4 Geometrical interpretation of the torsion vector

4.1 Infinitesimal covariant transports and torsion vector

Let a congruence of two parametric curves $x^i(\tau, \lambda)$ be given. Let the pair of vectors (u, ξ) be transported from point A with co-ordinates $x^i(\tau_0, \lambda_0)$ to point C with co-ordinates $x^i(\tau_0 + d\tau, \lambda_0)$ and on the other side, from point A with co-ordinates $x^i(\tau_0, \lambda_0)$ to point B with co-ordinates $x^i(\tau_0, \lambda_0 + d\lambda)$. The question arises under which conditions the vectors (u, ξ) , transported from point A to point C [where they will be the vectors $(\overline{u}, \overline{\xi})$] will be equal to the vectors (u, ξ) , transported to point B [where they will be the vectors $(\widetilde{u}, \widetilde{\xi})$].

The vectors \overline{u} and $\overline{\xi}$ at point C could be represented by means of the vectors u and ξ at point A if we use the exponent of the covariant differential operator. Then,

$$\overline{u} : = u_{(\tau_0+d\tau, \lambda_0)} = \left(\exp\left[d\tau \cdot \frac{D}{d\tau}\right] \right) u_{(\tau_0, \lambda_0)} \quad , \quad (133)$$

$$\overline{\xi} : = \xi_{(\tau_0+d\tau, \lambda_0)} = \left(\exp\left[d\tau \cdot \frac{D}{d\tau}\right] \right) \xi_{(\tau_0, \lambda_0)} \quad . \quad (134)$$

At the same time, the vectors \widetilde{u} and $\widetilde{\xi}$ at point C could also be represented by means of the vectors u and ξ at the point A in the form:

$$\widetilde{u} : = u_{(\tau_0, \lambda_0+d\lambda)} = \left(\exp\left[d\lambda \cdot \frac{D}{d\lambda}\right] \right) u_{(\tau_0, \lambda_0)} \quad , \quad (135)$$

$$\widetilde{\xi} : = \xi_{(\tau_0, \lambda_0+d\lambda)} = \left(\exp\left[d\lambda \cdot \frac{D}{d\lambda}\right] \right) \xi_{(\tau_0, \lambda_0)} \quad . \quad (136)$$

If the conditions

$$\overline{u} = \tilde{u} \quad , \quad \overline{\xi} = \tilde{\xi}$$

should be fulfilled for $d\tau = d\lambda = \varepsilon \ll 1$, then the relations

$$\left(\exp\left[\varepsilon \cdot \frac{D}{d\tau}\right] \right) u_{(\tau_0, \lambda_0)} = \left(\exp\left[\varepsilon \cdot \frac{D}{d\lambda}\right] \right) u_{(\tau_0, \lambda_0)} \quad , \quad (137)$$

$$\left(\exp\left[\varepsilon \cdot \frac{D}{d\tau}\right] \right) \xi_{(\tau_0, \lambda_0)} = \left(\exp\left[\varepsilon \cdot \frac{D}{d\lambda}\right] \right) \xi_{(\tau_0, \lambda_0)} \quad , \quad (138)$$

should be valid. The last conditions could be written in their explicit forms

$$\begin{aligned} & \left[\left(1 + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \dots \right) u \right]_{(\tau_0, \lambda_0)} \\ &= \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) u \right]_{(\tau_0, \lambda_0)} \quad , \end{aligned} \quad (139)$$

$$\begin{aligned} & \left[\left(1 + \varepsilon \cdot \frac{D}{d\tau} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\tau^2} + \dots \right) \xi \right]_{(\tau_0, \lambda_0)} \\ &= \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) \xi \right]_{(\tau_0, \lambda_0)} \quad . \end{aligned} \quad (140)$$

Up to the first order of ε , we obtain the relations

$$\left(\varepsilon \cdot \frac{Du}{d\tau} \right)_{(\tau_0, \lambda_0)} = \left(\varepsilon \cdot \frac{Du}{d\lambda} \right)_{(\tau_0, \lambda_0)} \quad , \quad \varepsilon \neq 0 \quad , \quad (141)$$

$$\left(\varepsilon \cdot \frac{D\xi}{d\tau} \right)_{(\tau_0, \lambda_0)} = \left(\varepsilon \cdot \frac{D\xi}{d\lambda} \right)_{(\tau_0, \lambda_0)} \quad , \quad \varepsilon \neq 0 \quad , \quad (142)$$

leading to the conditions [at the point $x^i(\tau_0, \lambda_0)$]

$$\nabla_u u = \nabla_\xi u \quad , \quad u = \frac{d}{d\tau} = u^i \cdot \partial_i \quad , \quad u^i = \frac{dx^i}{d\tau} \quad , \quad u \in T(M) \quad (143)$$

$$\nabla_u \xi = \nabla_\xi \xi \quad , \quad \xi = \frac{d}{d\lambda} = \xi^i \cdot \partial_i \quad , \quad \xi^i = \frac{dx^i}{d\lambda} \quad , \quad \xi \in T(M) \quad . \quad (144)$$

On the other side, the vectors u and ξ fulfill the relation

$$\mathcal{L}_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) \quad .$$

Substituting $\nabla_\xi u$ and $\nabla_u \xi$ from the above conditions, we obtain

$$\mathcal{L}_\xi u = \nabla_u u - \nabla_\xi \xi - T(\xi, u) \quad . \quad (145)$$

If the vectors u and ξ are tangent vectors to the co-ordinate lines τ and λ correspondingly, then $\mathcal{L}_\xi u = 0$ and the relation

$$\nabla_u u - \nabla_\xi \xi = T(\xi, u) \quad , \quad T(\xi, u) = T_{ij}{}^k \cdot \xi^i \cdot u^j \cdot \partial_k \in T(M) \quad , \quad (146)$$

follows. This means that under the assumption of the equivalence of the pairs $(\overline{u}, \overline{\xi})$ and $(\tilde{u}, \tilde{\xi})$ [obtained as a result of the transport of the pair (u, ξ) from point A to points C and B respectively] the torsion vector $T(\xi, u)$ and the corresponding torsion tensor $T = T_{ij}{}^k \cdot dx^i \wedge dx^j \otimes \partial_k$ can be interpreted as a measure for the deviation of the transport of a vector u along itself from the transport of the vector ξ along itself. It should be stressed that all vectors are compared to each other at the beginning point A of the transport, i.e. they are expressed by means of the corresponding vectors, found at point A by the use of the exponent mapping (exponent of the covariant differential operator) acting on the vectors u and ξ at point A .

4.2 Infinitesimal covariant transport and Lie derivative

Let us now consider the conditions for transports under which the Lie derivative remains unchanged.

Let the Lie derivative $\mathcal{L}_\xi u$ be given at a point A with co-ordinates $x^i(\tau_0, \lambda_0)$

$$[\mathcal{L}_\xi u]_{(\tau_0, \lambda_0)} = [\xi, u]_{(\tau_0, \lambda_0)} \quad . \quad (147)$$

At point B , the transported vectors ξ and u will have the forms respectively:

$$\tilde{\xi} : = \xi_{(\tau_0, \lambda_0 + d\lambda)} = \left(\exp[d\lambda \cdot \frac{D}{d\lambda}] \right) \xi_{(\tau_0, \lambda_0)} \quad , \quad (148)$$

$$\tilde{u} : = u_{(\tau_0, \lambda_0 + d\lambda)} = \left(\exp[d\lambda \cdot \frac{D}{d\lambda}] \right) u_{(\tau_0, \lambda_0)} \quad . \quad (149)$$

The Lie derivative $\mathcal{L}_{\tilde{\xi}} \tilde{u}$ at the point B could be expressed as

$$[\tilde{\xi}, \tilde{u}] = \left[\left(\exp[d\lambda \cdot \frac{D}{d\lambda}] \right) \xi_{(\tau_0, \lambda_0)} , \left(\exp[d\lambda \cdot \frac{D}{d\lambda}] \right) u_{(\tau_0, \lambda_0)} \right] \quad . \quad (150)$$

If we use the explicit form of the exponent of the covariant differential operator for the case $d\tau = d\lambda = \varepsilon$ we can write $[\tilde{\xi}, \tilde{u}]$ in the form

$$\begin{aligned} [\tilde{\xi}, \tilde{u}] &= \left[\left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) \xi \right]_{(\tau_0, \lambda_0)} , \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) u \right]_{(\tau_0, \lambda_0)} \right] = \\ &= \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) \xi \right]_{(\tau_0, \lambda_0)} \\ &\quad \circ \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) u \right]_{(\tau_0, \lambda_0)} - \\ &\quad - \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) u \right]_{(\tau_0, \lambda_0)} \\ &\quad \circ \left[\left(1 + \varepsilon \cdot \frac{D}{d\lambda} + \frac{1}{2!} \cdot \varepsilon^2 \cdot \frac{D^2}{d\lambda^2} + \dots \right) \xi \right]_{(\tau_0, \lambda_0)} \quad . \quad (151) \end{aligned}$$

Up to the first order of ε , we obtain

$$\begin{aligned} [\tilde{\xi}, \tilde{u}] &\approx [\xi, u]_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left[\frac{D\xi}{d\lambda}, u \right]_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left[\xi, \frac{Du}{d\lambda} \right]_{(\tau_0, \lambda_0)} = \\ &= [\xi, u]_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left\{ \left[\frac{D\xi}{d\lambda}, u \right] + \left[\xi, \frac{Du}{d\lambda} \right] \right\}_{(\tau_0, \lambda_0)} \quad . \quad (152) \end{aligned}$$

Since $[\xi, u] = \mathcal{L}_\xi u$ is a vector, we have

$$\begin{aligned} (\nabla_\xi [\xi, u])_{(\tau_0, \lambda_0)} &= \left(\frac{D}{d\lambda} [\xi, u] \right)_{(\tau_0, \lambda_0)} = \lim_{\varepsilon \rightarrow 0} \frac{[\xi, u]_{(\tau_0, \lambda_0 + d\lambda)} - [\xi, u]_{(\tau_0, \lambda_0)}}{\varepsilon} = \\ &= \lim_{\varepsilon \rightarrow 0} \frac{[\tilde{\xi}, \tilde{u}] - [\xi, u]_{(\tau_0, \lambda_0)}}{\varepsilon} = \left\{ \left[\frac{D\xi}{d\lambda}, u \right] + \left[\xi, \frac{Du}{d\lambda} \right] \right\}_{(\tau_0, \lambda_0)} \quad (153) \end{aligned}$$

Therefore, for every point of the curve $x^i(\tau, \lambda)$ the relation

$$\begin{aligned} \frac{D}{d\lambda} [\xi, u] &= \left[\frac{D\xi}{d\lambda}, u \right] + \left[\xi, \frac{Du}{d\lambda} \right] \\ \nabla_\xi [\xi, u] &= [\nabla_\xi \xi, u] + [\xi, \nabla_\xi u] \quad , \quad \xi = \frac{d}{d\lambda} \quad , \quad (154) \end{aligned}$$

is valid. Now we can write the expression

$$([\xi, u])_{(\tau_0, \lambda_0 + \varepsilon)} = ([\xi, u])_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left(\frac{D}{d\lambda} [\xi, u] \right)_{(\tau_0, \lambda_0)} . \quad (155)$$

For every given pair of vector fields (v, w) , $v, w \in T(M)$ analogous relations are fulfilled

$$([v, w])_{(\tau_0, \lambda_0 + \varepsilon)} = ([v, w])_{(\tau_0, \lambda_0)} + \varepsilon \cdot \left(\frac{D}{d\lambda} [v, w] \right)_{(\tau_0, \lambda_0)} . \quad (156)$$

On the basis of the last relation the following proposition can be proved:

Proposition 3 *The necessary and sufficient condition for not changing of the Lie derivative of two vector fields v and w along a curve $x^i(\tau_0 = \text{const.}, \lambda)$ is the condition*

$$\frac{D}{d\lambda} [v, w] = 0 \quad \text{or} \quad \nabla_\xi \mathcal{L}_v w = 0, \quad \xi = \frac{d}{d\lambda} . \quad (157)$$

Therefore, the Lie derivative $\mathcal{L}_v w = [v, w]$ of a contravariant vector field $w \in T(M)$ along an other contravariant vector $v \in T(M)$ could not change by an infinitesimal covariant transport from a point A with co-ordinates $x^i(\tau_0, \lambda_0)$ of a curve $x^i(\tau = \tau_0, \lambda)$ to a point B with co-ordinates $x^i(\tau = \tau_0, \lambda_0 + \varepsilon)$ of the same curve. This means that the necessary and sufficient condition for not changing of the Lie derivative under an infinitesimal transport from one to an other point of a curve is the vanishing of the covariant derivative of the Lie derivative along the vector field tangential to every point of the curve.

Special case: $\mathcal{L}_v w = 0$, $v, w \in T(M)$, $v := d/d\rho$, $w := d/d\sigma$. In this special case the vector field v and w are tangent vector fields to the corresponding co-ordinate lines $x^i(\rho, \sigma = \text{const.})$ and $x^i(\rho, \sigma = \text{const.})$. The congruence of these co-ordinate lines should obey the relations

$$\nabla_v \mathcal{L}_v w = 0, \quad \nabla_w \mathcal{L}_v w = 0, \quad (158)$$

i.e. the contravariant vector field $\mathcal{L}_v w = [v, w]$ should be transported parallel to the corresponding co-ordinate line.

If $\mathcal{L}_v w = [v, w] = 0$ then $\nabla_v w - \nabla_w v = T(v, w)$. The torsion vector $T(v, w)$ determines the difference between $\nabla_v w$ and $\nabla_w v$. If one of the vector fields w or v is transported parallel to the other, i.e. if $\nabla_v w = 0$ or $\nabla_w v = 0$ correspondingly, then the torsion vector $T(v, w) \neq 0$ is an obstacle for the other vector field (v or w correspondingly) to be also transported parallel to w or v correspondingly.. Therefore, the torsion vector (and the torsion tensor) could be interpreted as a measure for a deviation of parallel transport of a vector field, when the other vector field is transported parallel to it and at the same time both the vector fields are tangent vector fields to a congruence of two parametric lines which could be used as co-ordinate lines in a differentiable manifold M . In other words, if two co-ordinate lines in a space with torsion have tangent vectors, where the first of which could be transported parallel to the second vector, then the second tangent vector could not be transported parallel to the first one if the torsion vector is different from zero. This fact could be used for description of physical situations, where only one of the vector fields could be transported parallel along a co-ordinate line. This could have influence on the kinematic characteristics of a flow.

5 Kinematic characteristics of a flow

Let us now consider a parametric representation of an n -parametric (n -dimensional) congruence (family) of curves with respect to an arbitrary co-ordinate system [33]

in an n -dimensional differentiable manifold M

$$x^i = x^i(\tau, \lambda^a), \quad i = 1, 2, \dots, n, \quad a = 1, 2, \dots, n-1, \quad (\dim M = n). \quad (159)$$

The parameters λ^a designate the matter elements (material points), τ is the parameter along the curves interpreted for $n = 4$ as the proper time along the world lines $x^i = x^i(\tau, \lambda^a = \text{const.})$. The curves with parameter τ could also be interpreted as the line's having at every of its points tangential vector identified with the velocity of a material point at this line's point. The set of all such lines is call a flow. A single line of a flow is called line's flow. The tangent vector to the curves $x^i = x^i(\tau, \lambda^a = \lambda_0^a = \text{const.})$

$$u := \frac{\partial}{\partial \tau} = \frac{d}{d\tau} = \frac{\partial x^i}{\partial \tau} \cdot \partial_i = \frac{dx^i}{d\tau} \cdot \partial_i = u^i \cdot \partial_i, \quad u^i = \frac{\partial x^i}{\partial \tau} = \frac{dx^i}{d\tau}, \quad (160)$$

is interpreted as the velocity of the material points in the media. The transformation $x^i : (\tau, \lambda^a) \rightarrow x^i(\tau, \lambda^a)$ corresponds to the transformation in classical mechanics from Lagrangian to Eulerian co-ordinates [33].

Let $\xi_{(a)}$ be vectors, tangential to the curves $x^i(\tau = \text{const.}, \lambda^a)$. The curves $x^i(\tau = \tau_0 = \text{const.}, \lambda^a)$ describe the positions of material points for a given parameter (proper time) $\tau = \tau_0 = \text{const.}$, i.e.

$$\begin{aligned} \xi_{(a)} &:= \frac{\partial}{\partial \lambda^a} = \frac{d}{d\lambda^a} = \frac{\partial x^i}{\partial \lambda^a} \cdot \partial_i = \xi_{(a)}^i \cdot \partial_i, \quad \xi_{(a)}^i := \frac{\partial x^i}{\partial \lambda^a} = \frac{dx^i}{d\lambda^a} \quad (161) \\ \frac{d\lambda^a}{d\tau} &= 0, \quad \frac{d\tau}{d\lambda^a} = 0. \end{aligned}$$

If we consider a neighborhood of a point P of the flow with co-ordinates $x_P^i = x^i(\tau_0, \lambda_0^a)$ then we can find the co-ordinates of two points from a neighborhood of the point P denoted as point P_1 with co-ordinates $x_{P_1}^i = x^i(\tau = \tau_0 + d\tau, \lambda_0^a)$ and point P_2 with co-ordinates $x_{P_2}^i = x^i(\tau = \tau_0, \lambda^a = \lambda_0^a + d\lambda)$, where $\tau_0 = \text{const.}$, $\lambda_0 = \text{const.}$:

$$\begin{aligned} x_{(\tau_0+d\tau, \lambda_0^a)}^i &= x_{(\tau_0, \lambda_0^a)}^i + d\tau \cdot \left(\frac{\partial x^i}{\partial \tau} \right)_{(\tau_0, \lambda_0^a)} + O(d\tau^2) \approx \\ &\approx x_{(\tau_0, \lambda_0^a)}^i + d\tau \cdot u_{(\tau_0, \lambda_0^a)}^i = x_{(\tau_0, \lambda_0^a)}^i + \bar{u}_{(\tau_0, \lambda_0^a)}^i, \\ \bar{u}_{(\tau_0, \lambda_0^a)}^i &:= d\tau \cdot u_{(\tau_0, \lambda_0^a)}^i, \\ x_{(\tau_0, \lambda_0^a+d\lambda)}^i &= x_{(\tau_0, \lambda_0^a)}^i + d\lambda^a \cdot \left(\frac{\partial x^i}{\partial \lambda^a} \right)_{(\tau_0, \lambda_0^a)} + O((d\lambda^a)^2) \approx \\ &\approx x_{(\tau_0, \lambda_0^a)}^i + d\lambda^a \cdot \xi_{(a)(\tau_0, \lambda_0)}^i = x_{(\tau_0, \lambda_0^a)}^i + \bar{\xi}_{(a)(\tau_0, \lambda_0)}^i, \\ \bar{\xi}_{(a)(\tau_0, \lambda_0)}^i &:= d\lambda^a \cdot \xi_{(a)(\tau_0, \lambda_0)}^i, \\ &\quad (\text{there is no summation over } a). \end{aligned} \quad (162)$$

If the motion of a material point is described along the curve $x^i(\tau, \lambda^a = \lambda_0^a)$ from the point P with co-ordinates $x^i(\tau_0, \lambda_0^a)$ to the point P_1 with the co-ordinates $x^i(\tau_0 + d\tau, \lambda_0^a)$ then the velocity of this material point along the curve with parameter τ will be

$$u^i = \lim_{d\tau \rightarrow 0} \frac{x^i(\tau_0 + d\tau, \lambda_0^a) - x^i(\tau_0, \lambda_0^a)}{d\tau}. \quad (163)$$

Correspondingly, the velocity of a material point moving from the point P [with co-ordinates $x^i(\tau_0, \lambda_0^a)$] to the point P_2 [with the co-ordinates $x^i(\tau_0, \lambda_0^a + d\lambda^a)$] along the curve with parameter λ will be

$$\xi_{(a)}^i = \lim_{d\lambda \rightarrow 0} \frac{x^i(\tau_0, \lambda_0^a + d\lambda^a) - x^i(\tau_0, \lambda_0^a)}{d\lambda}. \quad (164)$$

Therefore, the vectors u and $\xi_{(a)}$ represent the velocity of material points moving on a curve with parameter τ or on a curve with parameter λ^a correspondingly. The notion of velocity along a curve is not yet physically interpreted. If the parameter τ is interpreted as the proper time of a material point moving on the curve $x^i(\tau, \lambda_0^a = \text{const.})$ then the vector $u = u^i \cdot \partial_i$ is its local velocity and the vector $\bar{\xi}_{(a)(\tau, \lambda_0^a + d\lambda^a)} = d\lambda^a \cdot \xi_{(a)(\tau, \lambda_0^a + d\lambda^a)}$ (there is no summation over a) is a distance vector of a material point from the point \bar{P} with co-ordinates $x^i(\tau, \lambda_0^a)$.

If τ and λ^a are chosen to be co-ordinates of the material points of a flow [proper frame of reference (moving with the points of the flow)] then the tangent vectors u and $\xi_{(a)}$ should obey the conditions [40]:

$$\mathcal{L}_u \xi_{(a)} = 0, \quad a = 1, \dots, n-1. \quad (165)$$

At the same time, the vectors $\xi_{(a)}$, $a = 1, \dots, n-1$, should be orthogonal to the vector u (in order to be a linear independent vector set in the manifold M with $\dim M = n$), i.e. $\xi_{(a)}$ should obey the relations

$$g(u, \xi_{(a)}) = 0. \quad (166)$$

The last conditions mean that $\xi_{(a)}$ could be expressed in the form

$$\xi_{(a)} = \xi_{(a)\perp} := \bar{g}[h_u(\xi_{(a)})], \quad (167)$$

because of the decomposition of the vectors $\xi_{(a)}$ in a part collinear to the vector u and a part orthogonal to u , i.e.

$$\xi_{(a)} = \frac{l_a}{e} \cdot u + \xi_{(a)\perp}, \quad (168)$$

where

$$\begin{aligned} l_a &:= g(u, \xi_{(a)}), \quad e := g(u, u) \neq 0, \quad h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u) \\ g &= g_{ij} \cdot dx^i \cdot dx^j, \quad g_{ij} = g_{ji}, \quad \bar{g} = g^{ij} \cdot \partial_i \cdot \partial_j, \quad g^{ij} = g^{ji}. \end{aligned} \quad (169)$$

For $l_a := 0$, we have $\xi_{(a)} = \xi_{(a)\perp}$.

6 Conclusion

In the present paper we have recall some basic notions and mathematical tools used in the structure of continuous media mechanics in Euclidean E_n ($n = 3$), and (pseudo) Riemannian spaces V_n ($n = 4$). Some of the notions are generalized for (\bar{L}_n, g) -spaces. The presented results are needed for the building out of continuous media mechanics in spaces with contravariant and covariant affine connections and metrics. It should be stressed that if we wish to describe a flow of material points (elements) we could take into account the velocities u and $\xi_{(a)}$ of the points along the corresponding curves together with their kinematic characteristics (such as relative velocity and acceleration, shear, rotation, and expansion velocities and accelerations) which consideration are coming next.

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